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THE OUTPUT AUTOCORRELATION FUNCTION AND POWER SPECTRAL DENSITY OF A DIODE FREQUENCY CONVERTER

BY
LOUIS J. IPPOLITO

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**THE OUTPUT AUTOCORRELATION FUNCTION
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OF A DIODE FREQUENCY CONVERTER**

**by
Louis J. Ippolito
Communications Research Branch**

March 1966

**GODDARD SPACE FLIGHT CENTER
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PREFACE

This document is based on a thesis submitted by the author to the Faculty of The George Washington University, School of Engineering and Applied Science. It is felt that the scope of the subject matter and its application to high-frequency communication systems studies warrants further publication and distribution by NASA/GSFC above that afforded by the recipient University.

ABSTRACT

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The output frequency spectrum of a diode frequency converter, or mixer, will contain sum and difference frequencies of the applied signal and local oscillator and their harmonics. In addition, these newly created frequencies will beat with each other and with the originally applied signals to create still more frequencies, and the process continues indefinitely. Advance knowledge of the location of the undesired frequencies is important to the design and performance characteristics of the converter, and to the sensitivity and filtering requirements of the receiver system.

This paper presents the output power spectral density of the diode frequency converter when subjected to deterministic, statistical, and mixed inputs by application of the Wiener-Khintchine Theorem to the autocorrelation function of the diode output current. The converter output autocorrelation function for the general signal plus noise case and for particular signal inputs is presented in terms of the diode conductance constants and the statistical moments of the input signal.

All of the autocorrelation and power spectra functions developed in this paper are expressed in series form to permit a useful analysis of the contributions of the diode conductance constants to each term. From these spectral displays, the optimum diode characteristic for efficient frequency conversion can be generated and the filtering and power requirements of the converter determined.

Author

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THE OUTPUT AUTOCORRELATION FUNCTION AND POWER SPECTRAL DENSITY OF A DIODE FREQUENCY CONVERTER

1. INTRODUCTION

Frequency conversion, or mixing, may be broadly defined as the conversion of a signal from one frequency band to another by combining it with a local oscillator voltage in a non-linear device. In a microwave communications system the frequency converter is utilized to convert the high-frequency received signal to a lower intermediate frequency where frequency selective operations such as amplification or detection may be more readily accomplished.

The essential part of the frequency converter consists of a device whose impedance to the high-frequency signal input is a non-linear function of the applied voltage. If the applied voltage includes the signal and a sinusoidal 'beating', or local oscillator, the output of the device will contain new frequencies generated by the non-linear impedance. The desired output frequency band, usually the difference of the local oscillator and signal frequencies, is then chosen by a suitable frequency selection or filtering network at the converter output.

The relatively recent extension of communications systems to the upper microwave and lower millimeter wavelength portion of the electromagnetic spectrum has produced an added importance on the techniques for achieving frequency conversion. The extremely high frequencies involved prevent the use of vacuum tubes as converters because of the losses produced from electron transit-time effects. The point contact semiconductor diode, where electrode spacing is on the order of atomic dimensions, has been found suitable as a non-linear element in the microwave frequency range. With the utilization of very small contact points and careful packaging and matching techniques, the semiconductor diode has proved superior to other methods of frequency conversion, and it is now in general use in microwave systems.

In this paper the output spectra of a microwave diode frequency converter subjected to deterministic and random input signals will be investigated. The term 'frequency converter', rather than 'mixer', is used here to stress the fact that this analysis is not limited to devices which translate the high-frequency input spectrum to a much lower intermediate frequency band, (say 30 or 100 Megacycles), but includes devices where the desired output spectrum may occupy a frequency band close to that of the input spectrum. The term mixer is usually associated with the former type of device, while the term frequency converter more accurately describes the latter.

1.1 SCOPE OF STUDY

The output spectrum of the diode frequency converter before preselection or filtering of the desired output frequencies will contain sum and difference frequencies of the applied signals and their harmonics. In addition, these newly created frequencies will beat with each other and with the originally applied signals to create still more frequencies, and the process continues indefinitely. Advance knowledge of the location of the undesired frequencies is important to the design and performance characteristics of the converter, and to the sensitivity and filtering requirements of the receiver system.

The output frequency spectrum of a converter subjected to an input signal which possesses completely deterministic characteristics can be predicted by a general analysis of the intermodulation products of the output current of the device¹. When the input signal is of a statistical or random nature, which is the prevalent condition in communications systems, the output spectrum is not directly available and the methods of statistical communications theory must be appealed to. The necessity of a statistical approach arises because it is usually impossible to specify the properties of the information bearing signal and the equally important noise that is present to sufficient accuracy without recourse to a probabilistic description of these processes.

This paper develops the output power spectral density of the diode frequency converter subjected to deterministic, random, and mixed inputs by application of the Wiener-Khinchine Theorem to the autocorrelation function of the diode output current. The general output autocorrelation function is presented in terms of the diode conductance constants and the moments of the input signal. The output statistics of the converter for particular inputs is also investigated and their output spectra displayed.

The paper is divided into eight sections. In Section 2 the mathematical preliminaries required for the analysis are summarized and results tabulated for later reference. In Section 3, the output current for the diode is derived by a Taylor Series analysis similar to Orloff's². Section 4 derives the general output autocorrelation function for the converter in terms of the statistical moments of the input and the diode conductance constants.

In Section 5 the statistical properties of the signal plus noise input are discussed and the general output autocorrelation function for the signal plus noise

¹See, for example; Orloff, (REF. 8), and Tucker, (REF. 16). The Reference List is found on page 90.

²Orloff, (REF. 8, pp. 173-175).

input is derived. The final three sections present solutions for particular deterministic and random inputs to the converter. In Section 6 the output power spectra for a cosine and a cosine plus narrowband noise input are presented. In Sections 7 and 8, results for amplitude and angle modulated carrier inputs are derived and their output spectra indicated.

Every attempt was made in this analysis to keep the resulting functions and expressions in a form which would allow a minimum amount of manipulation to obtain autocorrelation and power spectra directly. The basic approach guiding the analysis was to reduce the general autocorrelation function for the converter output current, $R_0(\tau)$, to a series of relatively simple functions of τ so that the resulting spectral contributions of each term were immediately available. This reduction was found to be most readily accomplished by expressing $R_0(\tau)$ in terms of the joint mixed moments of the input signal process, rather than in terms of a Bessel function factorization as done by Middleton³, or in terms of contour integrals of the input characteristic function, as done by Davenport and Root⁴. Shutterly derives series expressions for the output autocorrelation of the general non-linear device, but evaluation of his constants, requiring summation over five running integers, is unwieldy for all but the simplest of non-linear devices⁵.

The determination of the output statistics of the frequency converter when subjected to a cosine plus narrowband Gaussian noise input is included because it represents the most useful input for communications systems studies. As described in Sections 7 and 8, this input can be used to represent both the amplitude and the angle modulated cosine wave when subjected to a narrowband Gaussian modulating voltage.

³(REF. 7, Section 5.1).

⁴(REF. 3, Section 13.3).

⁵(REF. 12, Eqs. 38 and 66).

2. MATHEMATICAL PRELIMINARIES

This section summarizes some of the important mathematical and statistical concepts which will be utilized in the analysis of the succeeding sections, and indicates the nomenclature that will be followed in the text.

2.1 PROBABILITY DISTRIBUTIONS

The mathematical description of a random or stochastic process rests on the representation of the random mechanism as a set or ensemble of possible events possessing properties of statistical regularity. Consider a real, single random variable, x , the sample function of a random process, which has the range of values $-\infty \leq x \leq +\infty$. The probability that x is less than or equal to X , a particular value in the range of x , is the probability distribution function of x , written as $P(x \leq X)$. The extremal values of the probability distribution function are; $P(x \leq -\infty) = 0$, and $P(x \leq +\infty) = 1$, and it is a non-decreasing point function and everywhere continuous as $x \rightarrow +\infty$.

The probability density function, $p(x)$, is defined as the derivative of the probability distribution function

$$p(x) = \frac{d [P(x \leq X)]}{dX} \quad (2-1)$$

or

$$P(x \leq X) = \int_{-\infty}^x p(x) dx \quad (2-2)$$

From the extremal values of $P(x \leq X)$ it is apparent that

$$\int_{-\infty}^{+\infty} p(X) dx = 1 \quad (2-3)$$

and since $P(x \leq X)$ is a non-decreasing function, $p(x)$ is always ≥ 0 .

Any real function $y = g(x)$, of the real random variable x is itself a random variable with its own probability measures. The above functions can be extended to cover complex functions and are applicable to discrete, continuous, or mixed distributions. Consider now two random variables, x and y , which may be two separate one-dimensional variates or the components of the same sample space. For either condition a joint probability distribution can be defined by $P(x \leq X, y \leq Y)$, which represents the probability that the random variable x is less than or equal to a specified value X , and that y is less than or equal to the specified value Y .

The joint probability density function is the second mixed partial derivative,

$$p(x, y) = \frac{\partial^2}{\partial X \partial Y} P(x \leq X, y \leq Y)$$

or

$$P(x \leq X, y \leq Y) = \int_{-\infty}^y \int_{-\infty}^x p(x, y) dx dy \quad (2-4)$$

As in the one-dimensional case,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x, y) dx dy = 1 \quad (2-5)$$

2.2 TIME AND STATISTICAL AVERAGES

The time average of a sample function, $x(t)$, of a random process is

$$\langle x(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} x(t) dt \quad (2-6)$$

if the limit exists.

The statistical or ensemble average or expectation value of the sample function is

$$E[g(x_t)] = \int_{-\infty}^{+\infty} g(x_t) p(x_t) dx_t \quad (2-7)$$

where x_t refers to the possible values of $x(t)$ which can be assumed at the time t , and $p(x_t)$ is the probability density function associated with $x(t)$.

For an ergodic process, the time average equals the statistical average, with probability one. All of the random processes considered in this paper, whether due to random fluctuation noise or to random signals, will be assumed to possess the conditions necessary for ergodicity.

The n th moment of the probability distribution of $x(t)$ is

$$E[x^n(t)] = \int_{-\infty}^{+\infty} x^n(t) p(x) dx \quad (2-8)$$

The nth central moment is

$$E [(x - \bar{x})^n] = \int_{-\infty}^{+\infty} (x - \bar{x})^n p(x) dx \quad (2-9)$$

where \bar{x} is the mean value of x , $E[x]$.

A very important expectation value in statistical communication theory is the characteristic function of the probability distribution of $x(t)$, defined as

$$M_x(j\xi) = E[e^{j\xi x}] = \int_{-\infty}^{+\infty} e^{j\xi x} p(x) dx \quad (2-10)$$

The inverse Fourier Transform of $M_x(j\xi)$ is

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} M_x(j\xi) e^{-j\xi x} d\xi \quad (2-11)$$

The nth moment of $x(t)$ can be generated from its characteristic function by,

$$E[x^n(t)] = (-j)^n \left. \frac{d^n M_x(j\xi)}{d\xi^n} \right|_{\xi=0} \quad (2-12)$$

The above definitions can be extended to the case of multiple random variables by a suitable extension of the time and statistical average concepts. The joint moments of the joint probability distribution of the random variables x and y are defined as

$$E[x^n y^k] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^n y^k p(x, y) dx dy \quad (2-13)$$

The joint characteristic function is defined as

$$M_{xy}(j\xi_1, j\xi_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{(j\xi_1 x + j\xi_2 y)} p(x, y) dx dy \quad (2-14)$$

The joint characteristic function is the two-dimensional Fourier transform of the joint probability density function, i.e.,

$$p(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} M_{xy}(j\xi_1, j\xi_2) e^{-j\xi_1 x - j\xi_2 y} d\xi_1 d\xi_2 \quad (2-15)$$

The joint moments of x and y may be obtained from their joint characteristic function by,

$$E[x^n y^k] = (-j)^{n+k} \frac{\partial^{n+k} M_{xy}(j\xi_1, j\xi_2)}{\partial \xi_1^n \partial \xi_2^k} \bigg|_{\xi_1 = \xi_2 = 0} \quad (2-16)$$

Two random variables, x and y , are statistically independent when the mechanism producing x in no way affects y . The joint moment of two statistically independent variables reduces to

$$E[xy] = E[x] E[y] \quad (2-17)$$

2.3 CORRELATION FUNCTIONS

Let x_1 and x_2 be the random variables that refer to the possible values of the sample functions $x(t)$ of a given random process at the times t_1 and t_2 , respectively. The statistical autocorrelation function for that process is defined as the expectation value

$$R_x(t_1, t_2) = E[x_1 x_2] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1 x_2 p(x_1, x_2) dx_1 dx_2 \quad (2-19)$$

The time autocorrelation function of the sample function of a random process is defined as

$$R_x(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} x(t) x^*(t + \tau) dt \quad (2-20)$$

where $\tau = t_2 - t_1$, and the asterisk denotes the complex conjugate.

If the ensemble remains invariant under an arbitrary linear time shift, the process is stationary. For a stationary process, the statistical autocorrelation function of that process is a function only of the time difference τ . Therefore

$$R_x(t, t + \tau) = E[x_t, x_{t+\tau}^*] = R_x(\tau) \quad (2-21)$$

If the process is ergodic, time and statistical averages are the same, so

$$R_x(\tau) = \mathcal{R}_x(\tau) \quad (2-22)$$

Some random processes are not stationary in the strict sense, but nevertheless have invariant mean values and satisfy Eq. (2-21). Such random processes are said to be stationary in the wide sense.

The autocorrelation function of a stationary, real random process is an even function of its argument, while the crosscorrelation functions of two such processes may or may not be.

The following five sections will develop the correlation and crosscorrelation functions of the particular waveforms necessary for the frequency converter analysis.

Crosscorrelation of Cosines of Different Frequency

Consider the two cosine waves of constant amplitude and of radian frequencies ω_1 and ω_2 ,

$$f_1(t) = A_1 \cos(\omega_1 t + \phi_1)$$

$$f_2(t) = A_2 \cos(\omega_2 t + \phi_2)$$

The crosscorrelation of $f_1(t)$ and $f_2(t)$ is

$$R_{12}(\tau) = E[A_1 \cos(\omega_1 t + \phi_1) A_2 \cos(\omega_2 t + \omega_2 \tau + \phi_2)] \quad (2-23)$$

Since

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

therefore

$$\begin{aligned} R_{12}(\tau) &= \frac{A_1 A_2}{2} E[\cos(\omega_1 t + \phi_1 + \omega_2 t + \omega_2 \tau + \phi_2)] \\ &\quad + \frac{A_1 A_2}{2} E[\cos(\omega_1 t + \phi_1 - \omega_2 t - \omega_2 \tau - \phi_2)] \end{aligned}$$

or,

$$R_{12}(\tau) = \frac{A_1 A_2}{2} \left\{ \begin{aligned} &\cos(\phi_1 + \phi_2 + \omega_2 \tau) E[\cos(\omega_1 + \omega_2)t] \\ &- \sin(\phi_1 + \phi_2 + \omega_2 \tau) E[\sin(\omega_1 + \omega_2)t] \\ &+ \cos(\phi_2 - \phi_1 + \omega_2 \tau) E[\cos(\omega_2 - \omega_1)t] \\ &- \sin(\phi_2 - \phi_1 + \omega_2 \tau) E[\sin(\omega_2 - \omega_1)t] \end{aligned} \right\} \quad (2-24)$$

Since the expectation value of a sinusoid is zero,

$$R_{12}(\tau) = 0 \quad (2-25)$$

The Autocorrelation of a Cosine

Let

$$f_1(t) = f_2(t) = A_0 \cos(\omega_0 t + \phi)$$

Then from Eq. (2-24), the autocorrelation function will be

$$R_{11}(\tau) = \frac{A_0^2}{2} \left\{ \begin{aligned} &\cos(2\phi + \omega_0 \tau) E[\cos 2\omega_0 t] \\ &- \sin(2\phi + \omega_0 \tau) E[\sin 2\omega_0 t] + \cos \omega_0 \tau - 0 \end{aligned} \right\}$$

or

$$R_{11}(\tau) = \frac{A_0^2}{2} \cos \omega_0 \tau \quad (2-26)$$

Crosscorrelation of Cosines of Harmonic Frequency

Let

$$f_1(t) = A_1 \cos(n\omega_0 t + \phi_1)$$

$$f_2(t) = A_2 \cos(\omega_0 t + \phi_2)$$

where n is any integer > 1 .

Then $\omega_1 = n\omega_0$ and $\omega_2 = \omega_0$ in Eq. (2-23), and Eq. (2-24) becomes

$$\begin{aligned} R_{12}(\tau) = & \frac{A_1 A_2}{2} \cos(\phi_1 + \phi_2 + \omega_0 \tau) E[\cos(n+1)\omega_0 t] \\ & - \sin(\phi_1 + \phi_2 + \omega_0 \tau) E[\sin(n+1)\omega_0 t] \\ & + \cos(\phi_2 - \phi_1 + \omega_0 \tau) E[\cos(n-1)\omega_0 t] \\ & - \sin(\phi_2 - \phi_1 + \omega_0 \tau) E[-\sin(n-1)\omega_0 t] \end{aligned}$$

Since the expectation value of a sinusoid is zero,

$$R_{12}(\tau) = 0 \quad (2-27)$$

The Autocorrelation of a Cosine with Random Envelope and Phase

Let

$$f(t) = A(t) \cos[\omega_0 t + \phi(t)]$$

where $A(t)$ and $\phi(t)$ are ergodic, statistically independent random or mixed processes with $\phi(t)$ uniformly distributed over $0 \rightarrow 2\pi$.

The autocorrelation of $f(t)$ is therefore,

$$\begin{aligned} R_f(\tau) &= E\{f(t) f^*(t+\tau)\} \\ &= \frac{1}{2} \Re E \left\{ A(t) e^{j[\omega_0 t + \phi(t)]} A(t+\tau) e^{j[\omega_0 (t+\tau) + \phi(t+\tau)]} \right\} \\ &= \frac{1}{2} E \{ A(t) A(t+\tau) \} \Re E \left\{ e^{j[-\omega_0 \tau + \phi(t) - \phi(t+\tau)]} \right\} \\ &= \frac{1}{2} R_A(\tau) \Re E \left\{ e^{-j\omega_0 \tau} e^{j[\phi(t) - \phi(t+\tau)]} \right\} \end{aligned}$$

Therefore,

$$R_f(\tau) = \frac{1}{2} R_A(\tau) \cos \omega_0 \tau \Re E \left\{ e^{j[\phi(t) - \phi(t+\tau)]} \right\} \quad (2-30)$$

The properties of $\phi(t)$ must be further specified to determine the expectation value above. This is accomplished in Section 8.1 for the case of an angle modulated cosine wave.

The Autocorrelation of Narrowband Noise

Consider the band of "white" noise centered at a frequency f_0 cps, having a bandwidth of b cps and a power spectral density of a_0 watts/cps across the band. The two-sided spectrum representation, shown in Figure 2-1 (a), is

$$\begin{aligned} S(f) &= a_0 ; -\frac{b}{2} < f \pm f_0 < \frac{b}{2} \\ &= 0 ; \quad |f \pm f_0| > \frac{b}{2} \end{aligned} \quad (2-31)$$

The autocorrelation function is given by

$$\begin{aligned} R(\tau) &= \int_{\left(-f_0 - \frac{b}{2}\right)}^{\left(-f_0 + \frac{b}{2}\right)} a_0 e^{j 2 \pi f \tau} df + \int_{\left(f_0 - \frac{b}{2}\right)}^{\left(f_0 + \frac{b}{2}\right)} a_0 e^{j 2 \pi f \tau} df \\ &= \frac{a_0}{j 2 \pi \tau} \left[e^{j 2 \pi \frac{b}{2} \tau} - e^{-j 2 \pi \frac{b}{2} \tau} \right] e^{-j 2 \pi f_0 \tau} \\ &\quad + \frac{a_0}{j 2 \pi \tau} \left[e^{j 2 \pi \frac{b}{2} \tau} - e^{-j 2 \pi \frac{b}{2} \tau} \right] e^{j 2 \pi f_0 \tau} \end{aligned}$$

Re-arranging terms

$$\begin{aligned} R(\tau) &= \frac{2 a_0}{\pi \tau} \left[\frac{e^{j 2 \pi \frac{b}{2} \tau} - e^{-j 2 \pi \frac{b}{2} \tau}}{2 j} \right] \left[\frac{e^{j 2 \pi f_0 \tau} + e^{-j 2 \pi f_0 \tau}}{2} \right] \\ &= \frac{2 a_0}{\pi \tau} \sin 2 \pi \frac{b}{2} \tau \cos 2 \pi f_0 \tau \end{aligned}$$

or

$$R(\tau) = 2a_0 b \frac{\sin 2\pi \frac{b}{2} \tau}{2\pi \frac{b}{2} \tau} \cos 2\pi f_0 \tau \quad (2-32)$$

This function is a cosine wave at frequency f_0 , with a $\frac{\sin x}{x}$ envelope, and is shown in Figure 2-1(b). The autocorrelation function approaches zero for large values of τ , and has the value $2a_0 b$ at $\tau = 0$.

For $f_0 = 0$, corresponding to a low-pass band of frequencies $-\frac{b}{2} < f < \frac{b}{2}$, the carrier term in $R(\tau)$ disappears, and

$$R(\tau) = a_0 b \frac{\sin 2\pi \frac{b}{2} \tau}{2\pi \frac{b}{2} \tau} \quad (2-33)$$

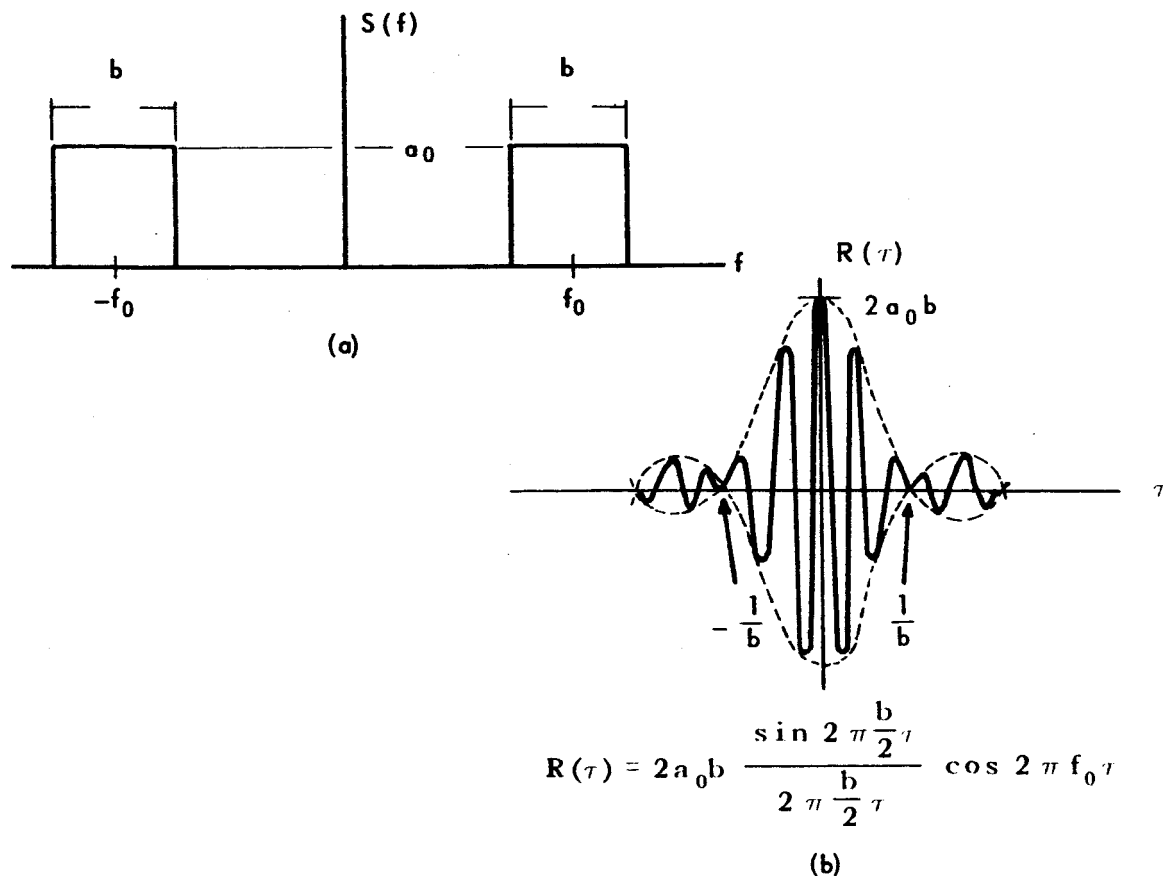


Figure 2-1. Autocorrelation Function of Narrowband White Noise

2.4 POWER SPECTRAL DENSITY

The total power associated with the random variable $x(t)$ is the time average of the total energy in $x(t)$, or

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 dt$$

The time autocorrelation function for $x(t)$ is, from Eq. (2-20),

$$R_x(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} x(t) x^*(t + \tau) dt$$

For $\tau = 0$,

$$R_x(0) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 dt$$

or

$$R_x(0) = P_x$$

The Fourier transform of $R_x(\tau)$ is defined as

$$S_x(f) = \int_{-\infty}^{+\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau$$

where $S_x(f)$ is called the power spectral density⁶.

The inverse Fourier transform of the above equation is

$$R_x(\tau) = \int_{-\infty}^{+\infty} S_x(f) e^{j2\pi f\tau} df \quad (2-34)$$

⁶ $S_x(f)$ has been termed the power density spectrum (by Lee), or the intensity spectrum (by Middleton), or simply the power spectrum (by Schwartz). The term power spectrum will occasionally be used here to refer to the graphical representation of $S_x(f)$ as a function of frequency.

For $\tau = 0$,

$$R_x(0) = \int_{-\infty}^{+\infty} S_x(f) df = P_x \quad (2-35)$$

The above relationship, which establishes the autocorrelation function and the power spectral density of a random function as a Fourier transform pair, is known as the Wiener-Khintchine Theorem for autocorrelation.

The autocorrelation function for a periodic function is periodic and the power spectral density is discrete. For a random function, the autocorrelation function is aperiodic and the power spectral density is continuous, with power contributed by a continuous band of frequencies.

The two-sided W-K transform is written as

$$R_x(\tau) = \int_{-\infty}^{+\infty} S_x(f) e^{j 2 \pi f \tau} df \quad (a)$$

(2-36)

$$S_x(f) = \int_{-\infty}^{+\infty} R_x(\tau) e^{-j 2 \pi f \tau} d\tau \quad (b)$$

Since $R_x(\tau)$ is real and an even function, $S_x(f)$ is likewise real and even, therefore the transform pair can be expressed as a pair of cosine transforms,

$$R_x(\tau) = \int_{-\infty}^{+\infty} S_x(f) \cos 2 \pi f \tau df \quad (a)$$

(2-37)

$$S_x(f) = \int_{-\infty}^{+\infty} R_x(\tau) \cos 2 \pi f \tau d\tau \quad (b)$$

The two-sided form of the transform pair, although introducing physically non-existent negative frequencies, is often useful because it allows the expression of series and integrals in exponential form, which may be easier to manipulate mathematically.

The two sections to follow will develop the power spectral density of a constant and a cosine wave from their respective autocorrelation functions, which are necessary for the converter analysis.

Power Spectral Density of a Constant

Let $x(t) = A_0$

then $R_x(\tau) = A_0^2$

From Eq. (2-37) (b),

$$S_x(f) = A_0^2 \int_{-\infty}^{+\infty} \cos 2\pi f \tau \, d\tau$$

As expected

$$S_x(f) = A_0^2 \delta(f - 0) \quad (2-38)$$

where $\delta(x)$ is the Dirac Delta Function.

Power Spectral Density of a Cosine Wave

Let $x(t) = A_0 \cos(\omega_0 t + \phi)$

then

$$R_x(\tau) = \frac{A_0^2}{2} \cos \omega_0 \tau$$

From Eq. (2-37) (b),

$$S_x(f) = \frac{A_0^2}{2} \int_{-\infty}^{+\infty} \cos \omega_0 \tau \cos \omega \tau \, d\tau$$

Hence

$$S_x(f) = \frac{A_0^2}{4} \int_{-\infty}^{+\infty} \left[\cos 2\pi (f + f_0) \tau + \cos 2\pi (f - f_0) \tau \right] d\tau$$

or

$$S_x(f) = \frac{A_0^2}{4} \left[\delta(f - f_0) + \delta(f + f_0) \right] \quad (2-39)$$

Each spectral 'line' of $S_x(f)$ has a magnitude of $A_0^2/4$, hence the total intensity is, as expected, equal to $A_0^2/2$.

2.5 PRODUCT TRANSFORMS AND CONVOLUTION

The determination of the power spectral density of the frequency converter from its autocorrelation function will involve the Fourier transformation of terms of the form $F[R_x(\tau)R_y(\tau)]$ and $F[R_x^2(\tau)]$. The resulting spectrum for these terms can be determined by application of the convolution integral of two functions

$$f(t) * g(t) = \int_{-\infty}^{+\infty} f(t - \tau) g(\tau) d\tau = \int_{-\infty}^{+\infty} f(\tau) g(t + \tau) d\tau \quad (2-40)$$

where the asterisk denotes a convolution.

For the two independent random variables x and y , with autocorrelations of $R_x(\tau)$ and $R_y(\tau)$ respectively, the spectral density of their product is given by,

$$S_{xy}(f) = F[R_x(\tau)R_y(\tau)] = \int_{-\infty}^{+\infty} R_x(\tau)R_y(\tau) e^{-j2\pi f\tau} d\tau$$

Since

$$R_x(\tau) = \int_{-\infty}^{+\infty} S_x(f) e^{j2\pi f\tau} df$$

Therefore

$$S_{xy}(f) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} S_x(f) e^{j2\pi f\tau} df \right] R_y(\tau) e^{-j2\pi f\tau} d\tau$$

Then let

$$S_{xy}(f) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} S_x(\xi) e^{j 2 \pi \xi \tau} d\xi \right] R_y(\tau) e^{-j 2 \pi f \tau} d\tau$$

where the new dummy variable ξ has been introduced to avoid possible confusion.

Hence

$$\begin{aligned} S_{xy}(f) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S_x(\xi) R_y(\tau) e^{-j 2 \pi (f - \xi) \tau} d\tau d\xi \\ &= \int_{-\infty}^{+\infty} S_x(\xi) d\xi \int_{-\infty}^{+\infty} R_y(\tau) e^{-j 2 \pi (f - \xi) \tau} d\tau \end{aligned}$$

Since

$$\int_{-\infty}^{+\infty} R_y(\tau) e^{-j 2 \pi (f - \xi) \tau} d\tau = S_y(f - \xi)$$

therefore

$$S_{xy}(f) = \int_{-\infty}^{+\infty} S_x(\xi) S_y(f - \xi) d\xi$$

Comparing this equation with Eq. (2-40) gives,

$$S_{xy}(f) = S_x(f) * S_y(f) \quad (2-42)$$

or

$$F[R_x(\tau) R_y(\tau)] = F[R_x(\tau)] * F[R_y(\tau)] \quad (2-43)$$

Extending the same approach, it can be shown that

$$F[R_x^2(\tau)] = F[R_x(\tau)] * F[R_x(\tau)]$$

or

$$F[R_x^n(\tau)] = F[R_x(\tau)] \overset{n}{*} F[R_x(\tau)] \quad (2-44)$$

where $\overset{n}{*}$ denotes n-fold self-convolutions.

The first three self-convolutions of the power spectrum of a cosine wave, $v(t) = A_c \cos \omega_c t$, are shown in Figure 2-2, and the convolutions of the spectra of two cosine waves at f_c and f_0 are shown in Figure 2-3.

The multiple convolution of the rectangular spectra of uniformly distributed narrowband noise, pictured in Figure 2-1 (a), is required for the converter analysis. The n^{th} self-convolution of a rectangular distribution of the form,

$$S(f) = 1 \quad ; \quad 0 < f < 1 \\ = 0 \quad ; \quad |f| > 1$$

is given by⁷,

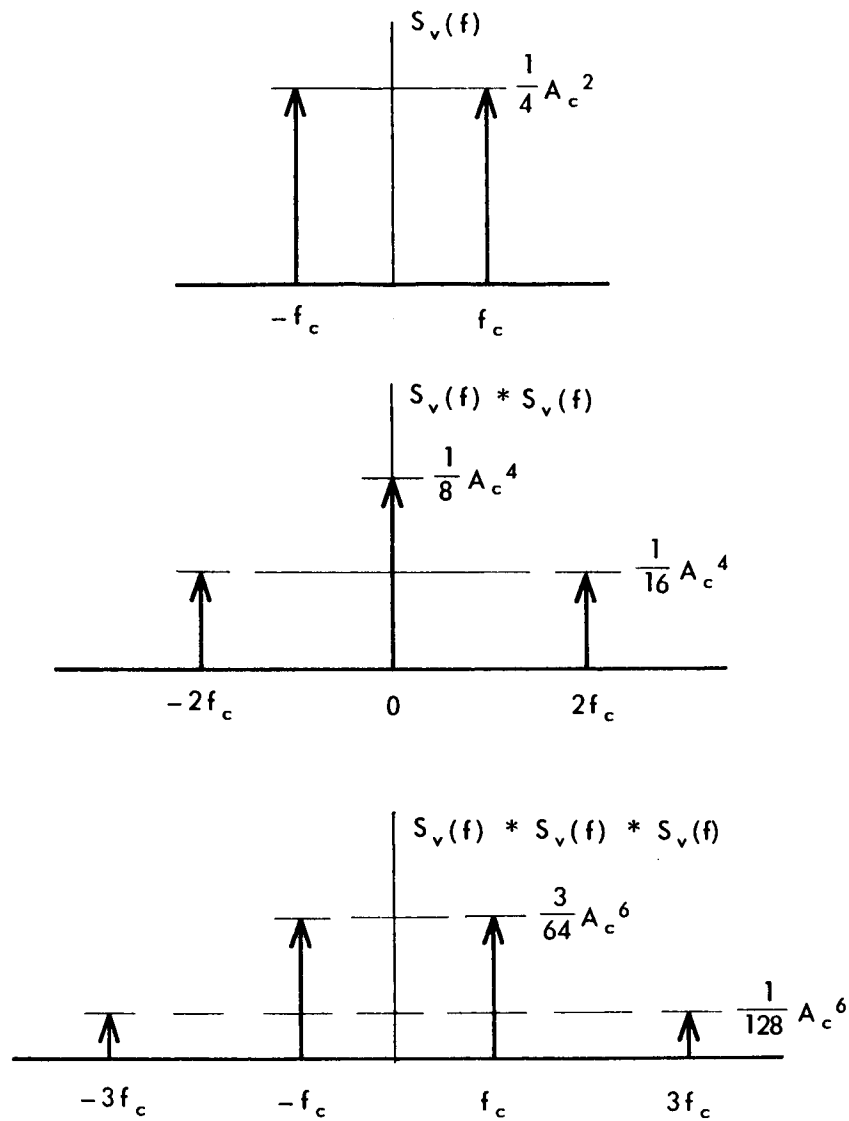
$$S(x) \overset{n}{*} S(x) = \frac{1}{(n-1)!} \left[x^{n-1} - \binom{n}{1} (x-1)^{n-1} + \binom{n}{2} (x-2)^{n-1} - \dots \right]$$

where $\binom{n}{j}$ denotes $j < x < n$, and the summation is continued as long as x , $(x-1)$, $(x-2)$, ... are positive.

The first three convolutions of the rectangular low-pass spectrum of bandwidth b and height a_0 are shown in Figure 2-4(a), and the first three convolutions of the rectangular spectrum centered at $\pm f_0$ are shown in Figure 2-4(b). As the number of convolutions is increased, the spectra of $S(f) \overset{n}{*} S(f)$ for the rectangle rapidly approaches a Gaussian shape, provided b remains finite.

The convolutions of the power spectra of a sinusoid with the first three self-convolutions of the rectangular narrowband noise are shown in Figure 2-5. The convolutions of the rectangular spectra with the self-convolutions of the sinusoid are shown in Figure 2-6.

⁷Cramer, (REF. 2, p. 245).



$$v(t) = A_c \cos \omega_c t$$

$$R_v(\tau) = \frac{A_c^2}{2} \cos \omega_c \tau$$

$$S_v(f) = \frac{A_c^2}{4} \delta(f \pm f_c)$$

Figure 2-2. Self-convolution of the Power Spectrum of a Sinusoid

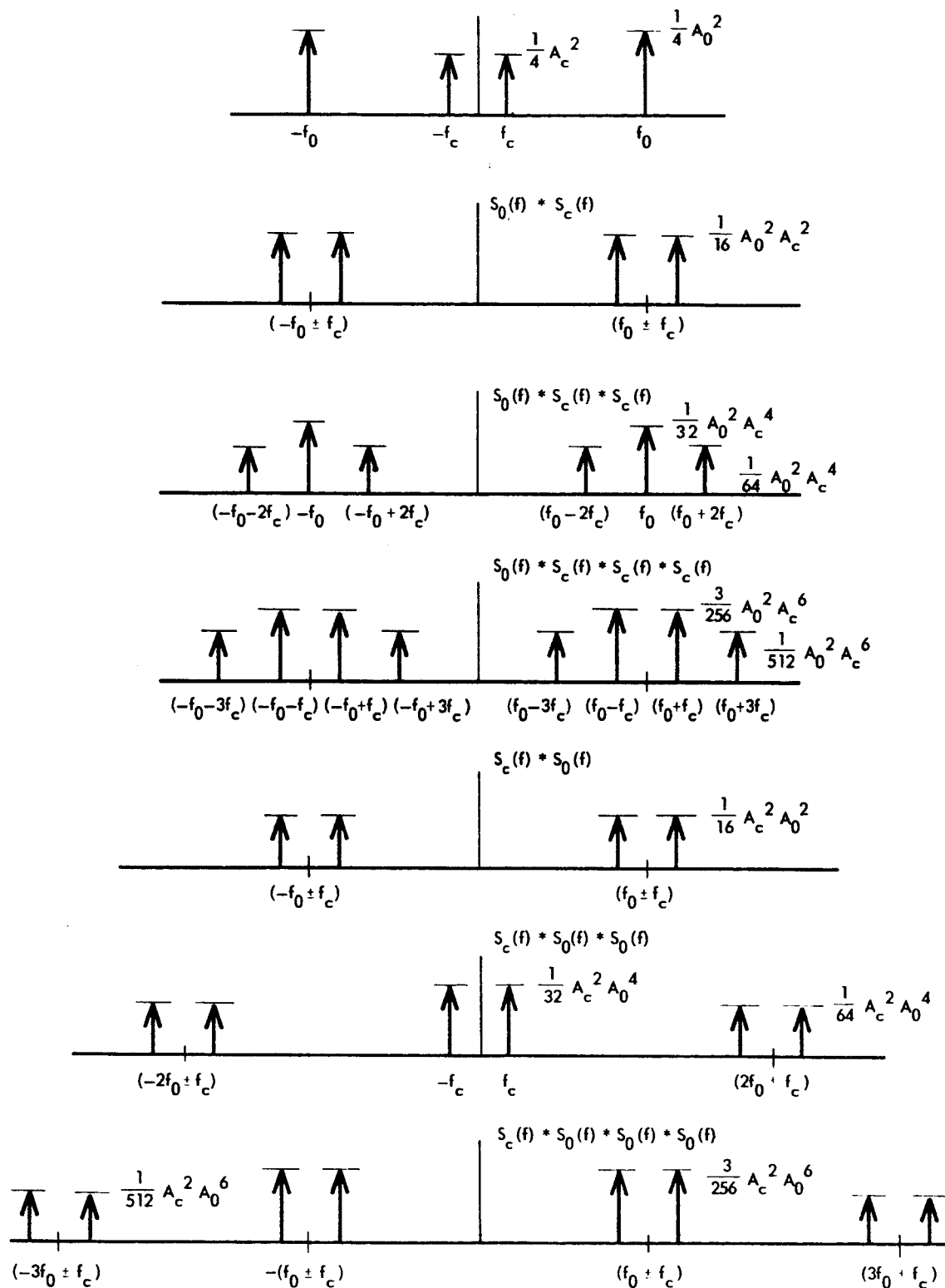


Figure 2-3. Convolution of the Power Spectra of Two Sinusoids

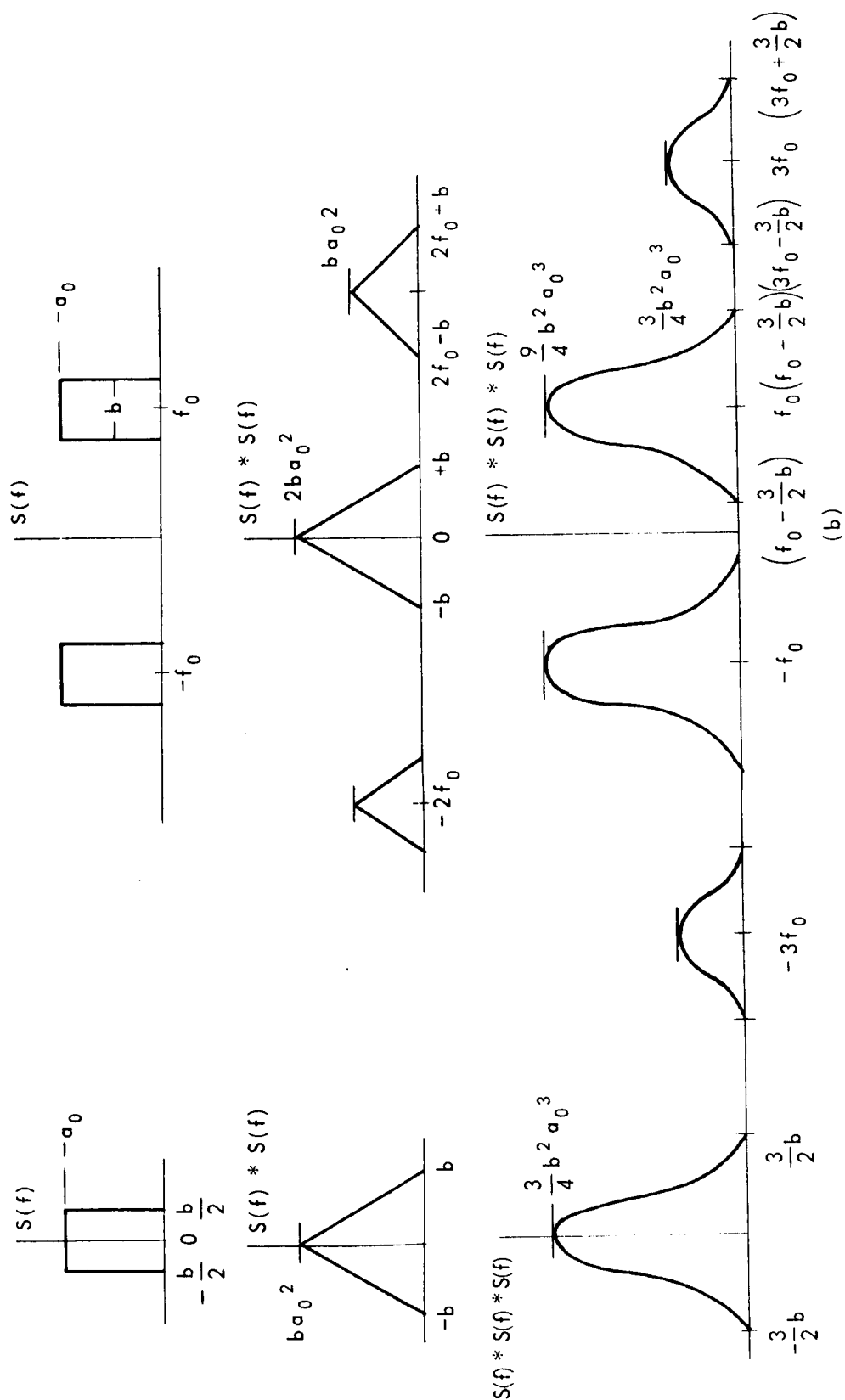


Figure 2-4. Self-convolution of Rectangular Power Spectra

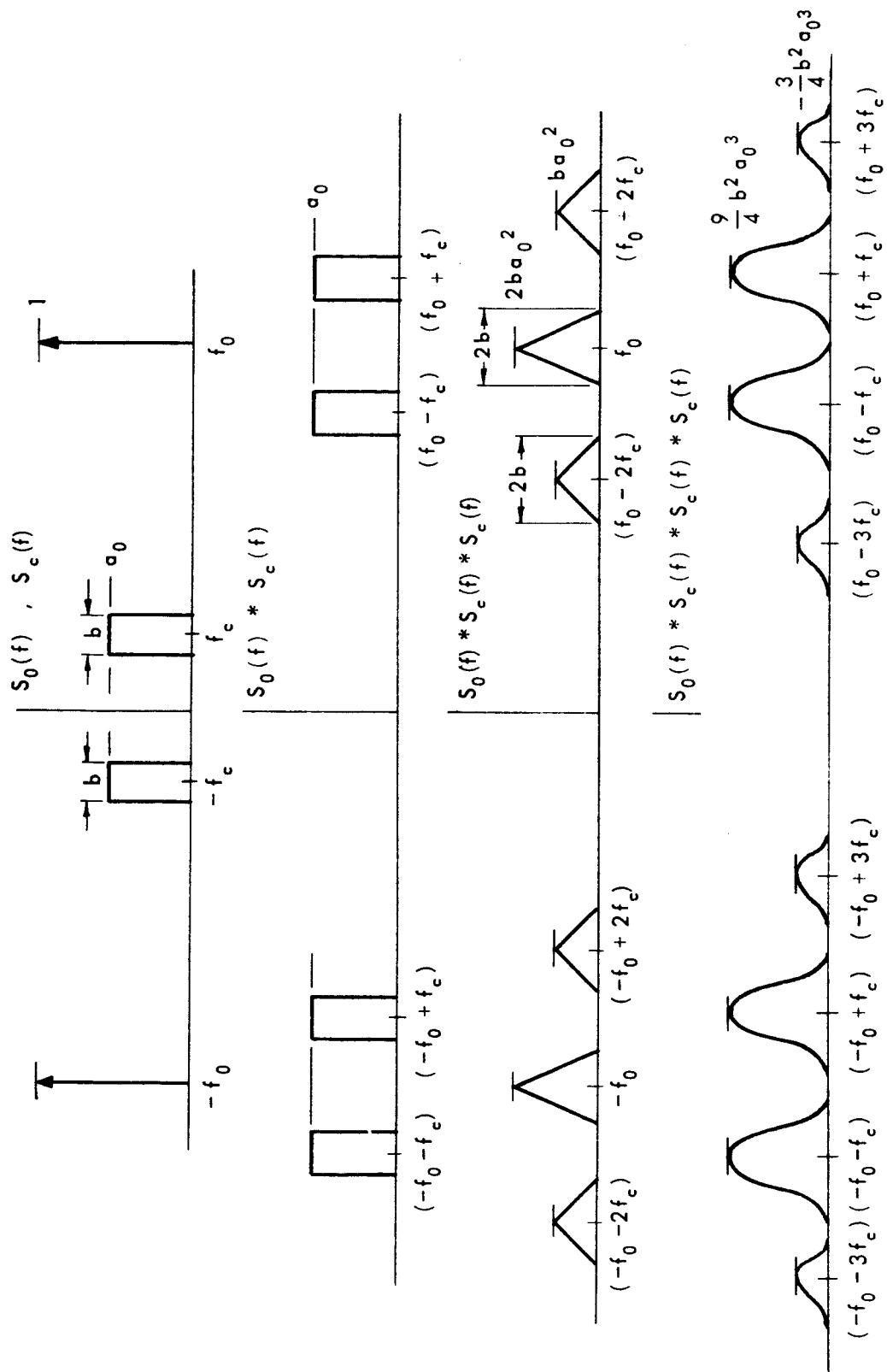


Figure 2-5. Convolution of the Power Spectra of a Sinusoid with the Self-convolutions of a Rectangular Spectra

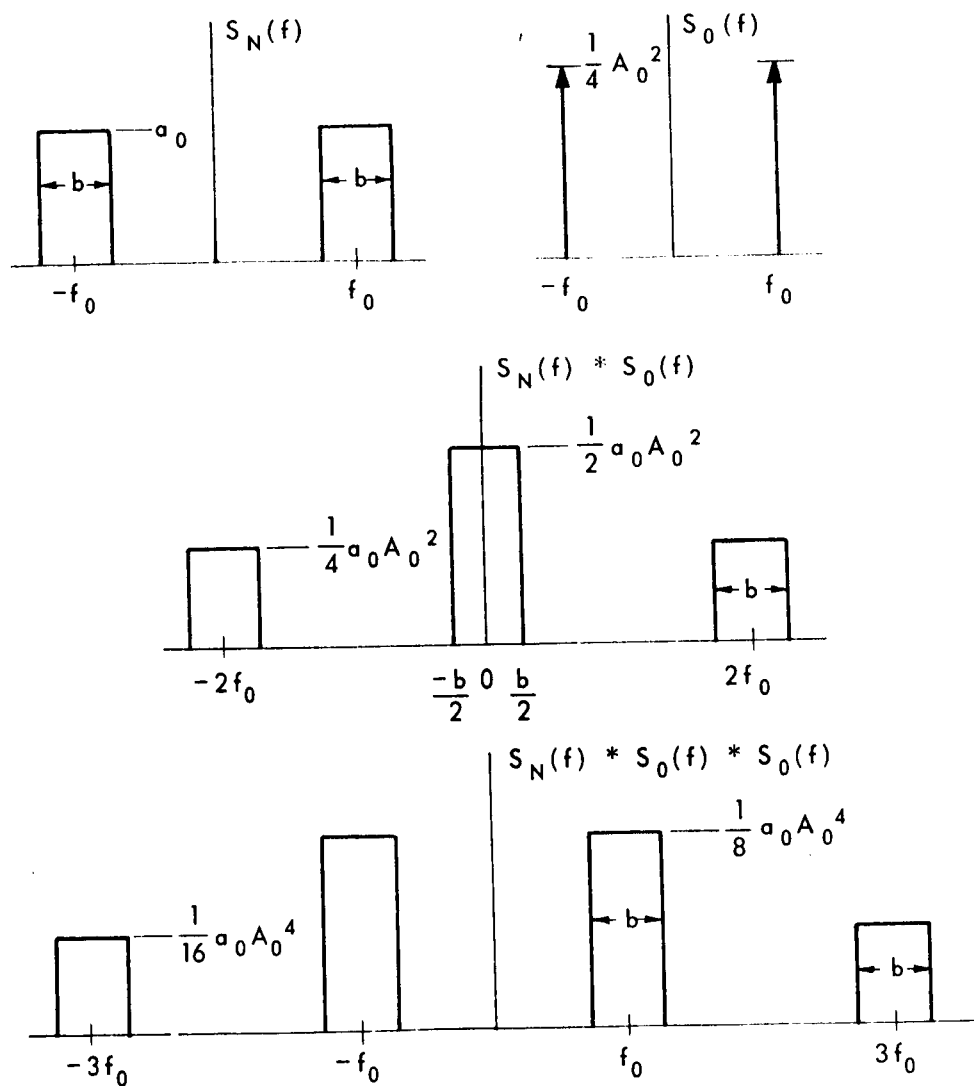


Figure 2-6. Convolution of the Rectangular Spectra with the Self-convolutions of a Sinusoid

3. CONVERTER OUTPUT CURRENT

The first quantitative theory describing the physical mechanisms of semiconductor rectification was developed independently in 1932 by Wilson, Nordheim, and others⁸, and was based on the quantum-mechanical 'tunnel' effect. Later work by Mott, Schottky, Bethe, and Herzfield led to the development of the diode and diffusion theories for semiconductor barrier behavior⁹. The fundamental relationship predicting the current-voltage characteristic of the variable resistance semiconductor diode has the form,

$$\begin{aligned} i_0(t) &= I_0 (e^{a v_d} - 1) & ; & |v| > 0 \\ &= 0 & ; & |v| < 0 \end{aligned} \quad (3-1)$$

where $i_0(t)$ is the current through the diode, v_d is the voltage applied across the barrier, and I_0 and a are constants for the particular diode of interest.

The high-frequency equivalent circuit for a semiconductor diode, which takes into account the known physical parameters at the junction, is shown in Figure 3-1(a). The circuit consists of a non-linear barrier resistance, R_b , in parallel with a barrier capacitance, C_b . The equivalent circuit for the diode frequency converter is shown in Figure 3-1(b).

The spreading resistance, r_s , results from the constriction of current flow in the semiconductor material near the contact. The magnitude of C_b is dependent on the applied voltage because the effective thickness of the barrier is a function of the applied voltage. This model does not include the effects of the diode cartridge and mounting configuration losses, which are assumed for the analysis to be minimized by proper design techniques.

The voltage applied to the diode, v , consists of the information-bearing signal voltage, $s(t)$, which may include undesired external noise components, and a sinusoidal local oscillator of the form,

$$v_0(t) = E_0 \cos \omega_0 t \quad (3-2)$$

The voltage across the barrier is,

$$v_d = v - i_0 R_s \quad (3-3)$$

⁸Torrey and Whitmer, (REF. 15, p. 77).

⁹ibid, p. 82.

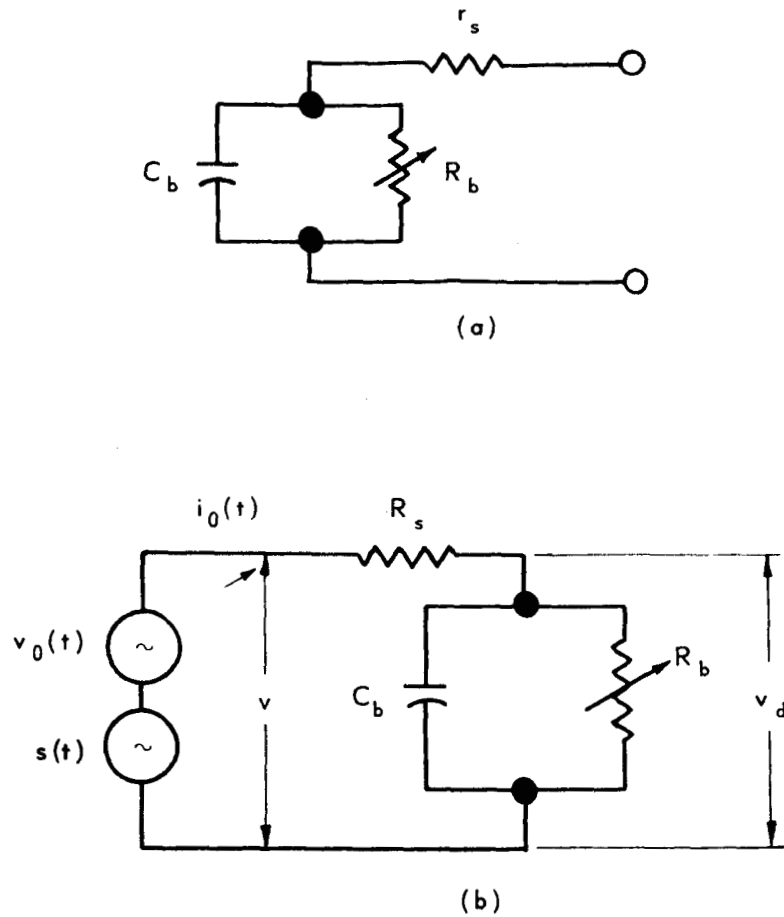


Figure 3-1. High-frequency Equivalent Circuits for (a) The Variable Resistance Semiconductor Diode and (b) The Diode Frequency Converter

where R_s is the sum of the diode spreading resistance and any series resistance from the voltage sources.

The exponential term in Eq. (3-1) can be expressed by its power series as,

$$e^{\alpha v_d} = 1 + \alpha v_d + \frac{(\alpha v_d)^2}{2!} + \frac{(\alpha v_d)^3}{3!} + \dots$$

Therefore,

$$i_0(t) = I_0 \left[\alpha v_d + \frac{(\alpha v_d)^2}{2!} + \frac{(\alpha v_d)^3}{3!} + \dots \right]$$

or

$$i_0(t) = \sum_{k=1}^{\infty} g_k v_d^k \quad (3-4)$$

where the conductance constants are given by

$$g_k = I_0 \frac{\alpha^k}{k!} = \frac{\alpha}{k} g_{k-1} \quad (3-5)$$

Neglecting R_s for the moment,

$$i_0(t) = \sum_{k=1}^{\infty} g_k v^k$$

If R_s is not neglected, the expressions for the g_k will be altered. The first three conductance constants with R_s included are derived in Appendix I, and are given by equations I-7, I-8, and I-9. For most microwave diodes of interest, $r_d \gg R_s$, and negligible error will be introduced by neglecting R_s . The validity of this assumption is further demonstrated by the calculations for the representative diode in Appendix II.

The diode current can therefore be expressed as

$$i_0(t) = \sum_{k=1}^{\infty} g_k [s(t) + v_0(t)]^k \quad (3-6)$$

For efficient frequency conversion, the so-called 'mixer condition' requires that

$$|v_0(t)| \gg |s(t)|$$

and for microwave receiver converters this is the prevalent condition. The diode current, because of the mixer condition, can be expanded in a Taylor Series about $v_0(t)$.

The Taylor expansion for a function, $f(x)$, about the point $x = a$ is,

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} \frac{d^n}{dx^n} f(a)$$

The diode current, from Eq. (3-6), is therefore,

$$i_0(t) = \sum_{n=0}^{\infty} \frac{[v - v_0(t)]^n}{n!} \frac{d^n}{dt^n} \left[\sum_{k=1}^{\infty} g_k v_0^k(t) \right] \quad (3-7)$$

Also

$$[v - v_0(t)]^n = [s(t) + v_0(t) - v_0(t)]^n = s^n(t)$$

hence,

$$i_0(t) = \sum_{n=0}^{\infty} \frac{s^n(t)}{n!} \frac{d^n}{dt^n} \left[\sum_{k=1}^{\infty} g_k v_0^k(t) \right] \quad (3-8)$$

The differential term in Eq. (3-8) can be expressed as a summation. Let,

$$I(t) = \sum_{k=1}^{\infty} g_k v_0^k(t)$$

Then

$$\frac{d^n}{dt^n} [I(t)] = \frac{d^n}{dt^n} [g_0 + g_1 v_0 + g_2 v_0^2 + \dots + g_k v_0^k + \dots]$$

$$\frac{d}{dt} [I(t)] = g_1 + 2g_2 v_0 + 3g_3 v_0^2 + \dots + k g_k v_0^{(k-1)} + \dots$$

$$\frac{d^2}{dt^2} [I(t)] = \dots + k(k-1) g_k v_0^{(k-2)} + \dots$$

·
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$$\frac{d^n}{dt^n} [I(t)] = \dots + k(k-1) \dots [k - (n-1)] g_k v_0^{(k-n)} + \dots$$

$$= \dots + \frac{k(k-1) \dots (k-n+1) (k-n) (k-n-1) \dots}{(k-n) (k-n-1) \dots} g_k v_0^{(k-n)} + \dots$$

or,

$$\frac{d^n}{dt^n} [I(t)] = \dots + \frac{k!}{(k-n)!} g_k v_0^{(k-n)} + \dots$$

Expressed as a series,

$$\frac{d^n}{dt^n} [I(t)] = \sum_{k=n}^{\infty} g_k \frac{k!}{(k-n)!} v_0^{(k-n)}(t) \quad (3-9)$$

Therefore Eq. (3-8) is,

$$i_0(t) = \sum_{n=0}^{\infty} \frac{s^n(t)}{n!} \sum_{k=n}^{\infty} g_k \frac{k!}{(k-n)!} v_0^{(k-n)}(t)$$

or,

$$i_0(t) = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{k!}{(k-n)! n!} g_k s^n(t) E_0^{(k-n)} \cos \omega_0 t^{(k-n)} \quad (3-10)$$

Let $k - n = p$, then $k = n + p$, and,

$$i_0(t) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(n+p)!}{p! n!} g_{(n+p)} E_0^k s^n(t) \cos^p \omega_0 t \quad (3-11)$$

Also let

$$K(n, p) = g_{(n+p)} E_0^p \frac{(n+p)!}{n! p!} \quad (3-12)$$

Therefore,

$$i_0(t) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} K(n, p) s^n(t) \cos^p \omega_0 t \quad (3-13)$$

In order to determine the form of the $i_0(t)$ terms, the above series will be expanded out to $n = 3$ and $p = 3$. This corresponds to letting k go to three in Eq. (3-4). Therefore,

$$\begin{aligned} i_0(t) = & K(0, 0) + K(0, 1) \cos \omega_0 t + K(0, 2) \cos^2 \omega_0 t \\ & + K(0, 3) \cos^3 \omega_0 t + K(1, 0) s(t) + K(1, 1) s(t) \cos \omega_0 t \\ & + K(1, 2) s(t) \cos^2 \omega_0 t + K(1, 3) s(t) \cos^3 \omega_0 t \\ & + K(2, 0) s^2(t) + K(2, 1) s^2(t) \cos \omega_0 t \\ & + K(2, 2) s^2(t) \cos^2 \omega_0 t + K(2, 3) s^2(t) \cos^3 \omega_0 t \\ & + K(3, 0) s^3(t) + K(3, 1) s^3(t) \cos \omega_0 t \\ & + K(3, 2) s^3(t) \cos^2 \omega_0 t + K(3, 3) s^3(t) \cos^3 \omega_0 t \end{aligned} \quad (3-14)$$

Using the identities,

$$\cos^2 \omega_0 t = \frac{1}{2} + \frac{1}{2} \cos 2\omega_0 t$$

$$\cos^3 \omega_0 t = \frac{3}{4} \cos \omega_0 t + \frac{1}{4} \cos 3\omega_0 t$$

Eq. (3-14) can be expressed as

$$\begin{aligned}
 i_0(t) = & A_{00} + A_{10}s(t) + A_{20}s^2(t) + A_{30}s^3(t) \\
 & + A_{01}\cos\omega_0 t + A_{02}\cos 2\omega_0 t + A_{03}\cos 3\omega_0 t \\
 & + A_{11}s(t)\cos\omega_0 t + A_{12}s(t)\cos 2\omega_0 t \\
 & + A_{13}s(t)\cos 3\omega_0 t + A_{21}s^2(t)\cos\omega_0 t \quad (3-15) \\
 & + A_{22}s^2(t)\cos 2\omega_0 t + A_{23}s^2(t)\cos 3\omega_0 t \\
 & + A_{31}s^3(t)\cos\omega_0 t + A_{32}s^3(t)\cos 2\omega_0 t \\
 & + A_{33}s^3(t)\cos 3\omega_0 t
 \end{aligned}$$

where,

$$\begin{aligned}
 A_{00} &= \left[K(0, 0) + \frac{1}{2} K(0, 2) \right] = g_0 E_0 + \frac{1}{2} g_2 E_0^2 \\
 A_{01} &= \left[K(0, 1) + \frac{3}{4} K(0, 3) \right] = g_1 E_0 + \frac{3}{4} g_3 E_0^3 \\
 A_{02} &= \frac{1}{2} K(0, 2) = \frac{1}{2} g_2 E_0^2 \quad (3-16) \\
 A_{03} &= \frac{1}{4} K(0, 3) = \frac{1}{4} g_3 E_0^3 \\
 A_{10} &= \left[K(1, 0) + \frac{1}{2} K(1, 2) \right] = g_1 + \frac{3}{2} g_3 E_0^2
 \end{aligned}$$

$$A_{11} = \left[K(1, 1) + \frac{3}{4} K(1, 3) \right] = 2 g_2 E_0 + 3 g_4 E_0^3$$

$$A_{12} = \frac{1}{2} K(1, 2) = \frac{3}{2} g_3 E_0^2$$

$$A_{13} = \frac{1}{4} K(1, 3) = g_4 E_0^3$$

$$A_{20} = \left[K(2, 0) + \frac{1}{2} K(2, 2) \right] = g_2 + 3 g_4 E_0^2$$

$$A_{21} = \left[K(2, 1) + \frac{3}{4} K(2, 3) \right] = g_3 E_0 + \frac{15}{2} g_5 E_0^3$$

$$A_{22} = \frac{1}{2} K(2, 2) = 3 g_4 E_0^2$$

(3-16)
contd.

$$A_{23} = \frac{1}{4} K(2, 3) = \frac{5}{2} g_5 E_0^3$$

$$A_{30} = \left[K(3, 0) + \frac{1}{2} K(3, 2) \right] = g_3 + \frac{5}{2} g_5 E_0^2$$

$$A_{31} = \left[K(3, 1) + \frac{3}{4} K(3, 3) \right] = 4 g_4 E_0 + 15 g_6 E_0^3$$

$$A_{32} = \frac{1}{2} K(3, 2) = 5 g_5 E_0^2$$

$$A_{33} = \frac{1}{4} K(3, 3) = 5 g_6 E_0^3$$

In general, therefore, the output current for the diode frequency converter will consist of a d.c. term, harmonics of the local oscillator voltage, and product terms of the signal and local oscillator. The output current can be expressed in series form as,

$$\begin{aligned}
 i_0(t) = & A_{00} \\
 & + \sum_{p=1}^{\infty} A_{0p} \cos p\omega_0 t \\
 & + \sum_{n=1}^{\infty} A_{n0} s^n(t) \\
 & + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} A_{np} s^n(t) \cos p\omega_0 t
 \end{aligned} \tag{3-17}$$

The A_{np} coefficients are functions of the diode conductance constants and of the local oscillator level, and can be obtained from Eq. (3-16).

4. CONVERTER OUTPUT AUTOCORRELATION FUNCTION

The autocorrelation function for the output current of the frequency converter is given by the expectation value,

$$R_0(\tau) = E \{ i_0(t) i_0(t + \tau) \} \quad (4-1)$$

where $i_0(t)$ is defined by Eq. (3-17). Substituting that equation into the above gives,

$$\begin{aligned} R_0(\tau) = & A_{00}^2 + E \left\{ 2 \sum_{p=1}^{\infty} \sum_{p'=1}^{\infty} A_{0p} A_{0p'} \cos p\omega_0 t \cos p'\omega_0(t + \tau) \right\} \\ & + E \left\{ 2 \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} A_{n0} A_{n'0} s^n(t) s^{n'}(t + \tau) \right\} \\ & + 2E \left\{ \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \sum_{p=1}^{\infty} \sum_{p'=1}^{\infty} A_{np} A_{n'p'} s^n(t) \cos p\omega_0 t s^{n'}(t + \tau) \cos p'\omega_0(t + \tau) \right\} \\ & + 2E \left\{ A_{00} \sum_{p=1}^{\infty} A_{0p} \cos p\omega_0 t \right\} + 2E \left\{ A_{00} \sum_{n=1}^{\infty} A_{n0} s^n(t) \right\} \\ & + 2E \left\{ A_{00} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} A_{np} s^n(t) \cos p\omega_0 t \right\} \\ & + 2E \left\{ \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} A_{0p} A_{n0} \cos p\omega_0 t s^n(t + \tau) \right\} \\ & + 2E \left\{ \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p'=1}^{\infty} A_{0p} A_{np'} s^n(t) \cos p\omega_0 t \cos p'\omega_0(t + \tau) \right\} \\ & + 2E \left\{ \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \sum_{p=1}^{\infty} A_{n0} A_{n'p} s^n(t) s^{n'}(t + \tau) \cos p\omega_0(t + \tau) \right\} \end{aligned}$$

The second term in Eq. (4-2) is a cross-correlation of two cosines, which is identically zero for all $p \neq p'$, (see Eq. 2-25), and is of the form

$$\frac{1}{2} \sum_{p=1}^{\infty} A_0^2 \cos p\omega_0 \tau$$

for all $p = p'$, (see Eq. 2-26).

The fourth term is of the form,

$$E \{ s^n(t) s^{n'}(t + \tau) \cos p\omega_0 t \cos p'\omega_0(t + \tau) \}$$

Since $s(t)$ and $v_0(t)$ are assumed to be statistically independent, the above expectation value can be expressed as (see Eq. 2-17),

$$E \{ s^n(t) s^{n'}(t + \tau) \} E \{ \cos p\omega_0 t \cos p'\omega_0(t + \tau) \}$$

The above expression will be zero for all $p \neq p'$, and will be

$$E \{ s^n(t) s^n(t + \tau) \} \cos p\omega_0 \tau$$

for all $p = p'$.

The fifth term is zero since,

$$E \{ \cos p\omega_0 t \} = 0$$

The seventh and eighth terms are likewise zero because

$$E \{ s^n(t) \cos p\omega_0 t \} = E \{ s^n(t) \} E \{ \cos p\omega_0 t \} = 0$$

The ninth term is zero for all $p \neq p'$, and is

$$E \{ s^n(t) \} \cos p\omega_0 \tau$$

for all $p = p'$.

The tenth term is zero since,

$$E \{ s^n(t) s^{n'}(t + \tau) \} E \{ \cos p\omega_0 t \} = 0$$

Introducing the following definition to express the mixed moment terms,

$$\Phi_{S^{nn'}}(\tau) = E \{ s^n(t) s^{n'}(t + \tau) \} \quad (4-3)$$

Eq. (4-2) reduces to,

$$\begin{aligned} R_0(\tau) = & A_{00}^2 + \frac{1}{2} \sum_{p=1}^{\infty} A_{0p}^2 \cos p\omega_0\tau + \sum_{n=1}^{\infty} A_{n0}^2 \Phi_{S^{nn}}(\tau) \\ & + 2 \sum_{n=1}^{\infty} \sum_{\substack{n'=1 \\ n \neq n'}}^{\infty} A_{n0} A_{n'0} \Phi_{S^{nn'}}(\tau) \\ & + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} A_{np}^2 \Phi_{S^{nn}}(\tau) \cos p\omega_0\tau \quad (4-4) \\ & + \sum_{n=1}^{\infty} \sum_{\substack{p=1 \\ n \neq n'}}^{\infty} \sum_{n'=1}^{\infty} A_{np} A_{n'p} \Phi_{S^{nn'}}(\tau) \cos p\omega_0\tau \\ & + 2A_{00} \sum_{n=1}^{\infty} A_{n0} \Phi_{S^{n0}}(\tau) \\ & + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} A_{0p} A_{np} \Phi_{S^{n0}}(\tau) \cos p\omega_0\tau \end{aligned}$$

Re-arranging the terms of Eq. (4-4) gives,

$$\begin{aligned} R_0(\tau) = & A_{00}^2 + \frac{1}{2} \sum_{p=1}^{\infty} A_{0p}^2 \cos p\omega_0\tau \\ & + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \left[2A_{00} A_{n0} + A_{0p} A_{np} \cos p\omega_0\tau \right] \Phi_{S^{n0}}(\tau) \quad (4-5) \end{aligned}$$

$$\begin{aligned}
& + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \left[A_{n0}^2 + \frac{1}{2} A_{np}^2 \cos p\omega_0 \tau \right] \Phi_{snn}(\tau) \\
& + \sum_{\substack{n=1 \\ n \neq n'}}^{\infty} \sum_{\substack{n'=1 \\ n' \neq n}}^{\infty} \sum_{p=1}^{\infty} \left[2 A_{n0} A_{n'0} + A_{np} A_{n'p} \cos p\omega_0 \tau \right] \Phi_{snn'}(\tau)
\end{aligned}
\tag{4-5}$$

contd

The equation above expresses the complete output autocorrelation function of the frequency converter in terms of the diode constants and the moments of the input signal. Once the statistical characteristics of $s(t)$ are known and the mixed moments are generated, $R_0(\tau)$ can be found and the resulting power spectral density described.

The following section will develop the statistical moments for an input consisting of a signal and additive noise, and will describe the resulting autocorrelation function.

5. SIGNAL PLUS NOISE INPUT CONDITION

For most practical communications systems the input to the frequency converter can be considered as a mixed statistical process consisting of the additive sum of a desired signal voltage, $v(t)$, and a non-deterministic noise voltage, $N(t)$, i.e.,

$$s(t) = v(t) + N(t) \quad (5-1)$$

The signal voltage consists of a cosine carrier of frequency f_c amplitude or angle modulated by a modulating voltage which is in some way proportional to the information being transmitted. The noise voltage is assumed to possess the fluctuation characteristics of shot and thermal noise, hence its properties can be described by the normal (Gaussian) statistical process.

5.1 STATISTICAL PROPERTIES OF $s(t)$

The general input voltage to the frequency converter, $s(t)$, can be represented as

$$s(t) = A_0 [1 + aA(t)] \cos [\omega_c t + \phi_0 + b\phi(t)] + cN(t) \quad (5-2)$$

The values of the constants, a , b , and c , will depend on the type of modulation present in the transmitted signal and on the presence of noise. For example, if the signal consists of noise alone, then $a = b = 0$, and $c = 1$, so,

$$s(t) = N(t)$$

If $s(t)$ is an amplitude modulated wave accompanied by noise, then $c = 1$ and $b = 0$, or

$$s(t) = A_0 [1 + aA(t)] \cos [\omega_c t + \phi_0] + N(t) \quad (5-3)$$

If $s(t)$ is an angle modulated waveform accompanied by noise, $a = 0$ and $c = 1$, so

$$s(t) = A_0 \cos [\omega_c t + \phi_0 + b\phi(t)] + N(t) \quad (5-4)$$

The statistical conditions on $v(t)$, $N(t)$, $A(t)$, and $\phi(t)$ that are assumed for this analysis are summarized below.

(1) $v(t)$ and $N(t)$ are statistically independent and are at least wide sense stationary.

(2) $A(t)$ and $\phi(t)$ are slowly varying compared to $\cos \omega_c t$, so that $v(t)$ is narrowband and $A(t)$ and $\phi(t)$ can be considered as modulation in the usual sense.

(3) $A(t)$ and $\phi(t)$ may be periodic, entirely random, or mixed processes.

(4) $A(t)$ and $\phi(t)$ may be statistically related, or correlated.

(5) $A(t)$ and $\phi(t)$ are at least wide sense stationary¹⁰.

(6) $A(t)$ and $\phi(t)$ are statistically independent of ϕ_0 , the carrier phase. This condition exists for most modulation impression techniques, where the modulation voltage is impressed on the carrier without regard to a fixed phase relation.

(7) $N(t)$ is described by a Gaussian statistical process with zero mean and a variance of σ_N^2 .

5.2 JOINT MOMENTS OF $s(t)$

The statistical properties required for the solution of $s(t)$ from Eq. (4-5) are the moments of $s(t)$, $\Phi_{s^{n0}}(\tau)$, and the higher order joint moments, $\Phi_{s^{nn}}(\tau)$ and $\Phi_{s^{nn}}(\tau)$. By successive differentiation of the characteristic function for the Gaussian distribution, the n th moment of $N(t)$ is found¹¹ to be zero for n even, and a multiple of the second moment, σ_N^2 , for n odd. That is,

$$\begin{aligned} E[N^n(t)] &= 0 && ; n \text{ odd} \\ &= 1 \cdot 3 \cdot 5 \dots (n-1) \sigma_N^n && ; n \geq 2 \text{ even} \end{aligned} \quad (5-5)$$

The moments of $s(t)$ are expressed as,

$$\Phi_{s^{n0}}(\tau) = E[s^n(t)] = E\{[v(t) + N(t)]^n\}$$

¹⁰ Even though $A(t)$ and $\phi(t)$ are stationary, $v(t)$ may not itself be so. It is necessary that the carrier frequency phase, ϕ_0 , be uniformly distributed in the primary interval, 0 to 2π , for $v(t)$ to also be stationary. This condition is discussed by Middleton, (REF. 6, Section 1.3-7).

¹¹ Davenport & Root, (REF. 3, p. 147).

For $|v(t)| > |N(t)|$, the binomial series expansion gives,

$$\begin{aligned}\Phi_{s^{n0}}(\tau) &= E \left\{ \sum_{l=0}^n \frac{n!}{(n-l)! l!} v(t)^{(n-l)} N(t)^l \right\} \\ &= \sum_{l=0}^n \frac{n!}{(n-l)! l!} E \{ v^{(n-l)}(t) \} E \{ N^l(t) \}\end{aligned}$$

From Eq. (5-5),

$$\Phi_{s^{n0}}(\tau) = \sum_{l=0,2,4}^n \frac{[1 \cdot 3 \cdot 5 \dots (n-l)] n! \sigma_N^n}{(n-l)! l!} E \{ v^{(n-l)}(t) \} \quad (5-6)$$

The joint moments for $s(t)$ are similarly expanded as,

$$\begin{aligned}\Phi_{s^{nn}}(\tau) &= E \{ s^n(t) s^n(t+\tau) \} \\ &= E \{ [v(t) + N(t)]^n [v(t+\tau) + N(t+\tau)]^n \}\end{aligned}$$

Applying the binomial series to the above expectation value gives,

$$\begin{aligned}\Phi_{s^{nn}}(\tau) &= \\ E \left\{ \left[\sum_{l=0}^n \frac{n!}{(n-l)! l!} v^{(n-l)}(t) N^l(t) \right] \left[\sum_{l'=0}^n \frac{n!}{(n-l')! l'!} v^{(n-l')}(t+\tau) N^{l'}(t+\tau) \right] \right\}\end{aligned}$$

or,

$$\Phi_{s^{nn}}(\tau) =$$

$$\sum_{l=0}^n \sum_{l'=0}^n \frac{(n!)^2}{(n-l)!(n-l')!l!l'} E\{v^{(n-l)}(t) v^{(n-l')}(t+\tau)\} E\{N^l(t) N^{l'}(t+\tau)\} \quad (5-7)$$

The joint mixed moments for $s(t)$ are given by,

$$\Phi_{s^{nn'}}(\tau) = E\{[v(t) + N(t)]^n [v(t+\tau) + N(t+\tau)]^{n'}\} \quad (n \neq n')$$

Applying the binomial series to the above gives,

$$\Phi_{s^{nn'}}(\tau) =$$

$$\sum_{l=0}^n \sum_{\substack{l'=0 \\ n \neq n'}}^{n'} \frac{n! n'!}{(n-l)!(n'-l')!l!l'} E\{v^{(n-l)}(t) v^{(n'-l')}(t+\tau)\} E\{N^l(t) N^{l'}(t+\tau)\} \quad (5-8)$$

In order to simplify the expressions for the moments of $s(t)$ somewhat, new running variables will be introduced into the expressions above. Let,

$$\alpha = n - l$$

$$\beta = n - l'$$

$$\gamma = n' - l'$$

$$\delta = n' - l'$$

The moments of $s(t)$ can then be expressed as,

$$\Phi_{S^{n0}}(\tau) = \sum_{l=0,2,4}^n \frac{[1 \cdot 3 \cdot 5 \dots (n-1)] n! \sigma_N^n}{\alpha! l!} \Phi_{v^{n0}}(\tau) \quad (5-9)$$

and the joint moments as,

$$\Phi_{S^{nn}}(\tau) = \sum_{l=0}^n \sum_{l'=0}^n \frac{(n!)^2}{\alpha! \beta! l! l'!} \Phi_{v^{\alpha\beta}}(\tau) \Phi_{N^{ll'}}(\tau) \quad (5-10)$$

and the joint mixed moments as,

$$\Phi_{S^{nn'}}_{n \neq n'}(\tau) = \sum_{l=0}^n \sum_{l'=0}^{n'} \frac{n! n'!}{\alpha! \gamma! l! l'!} \Phi_{v^{\alpha\delta}}(\tau) \Phi_{N^{ll'}}(\tau) \quad (5-11)$$

The moments, joint moments, and joint mixed moments of $s(t)$ from the above equations are listed in Table 5-1 for n and n' out to three. For this table,

$$E[v(t)] = 0$$

$$E[v^2(t)] = \sigma_v^2$$

$$E[v(t) v(t+\tau)] = R_v(\tau)$$

$$E[N(t)] = E[N^3(t)] = 0$$

$$E[N(t) N(t+\tau)] = R_N(\tau)$$

$$E[N^2(t)] = \sigma_N^2$$

Also, since both $v(t)$ and $N(t)$ are assumed at least wide sense stationary,

$$S(t) = v(t) + N(t)$$

$$\Phi_{S^{nn'}}(\tau) = E \{ S^n(t) S^{n'}(t + \tau) \}$$

$$\Phi_{S^{10}}(\tau) = 0$$

$$\Phi_{S^{20}}(\tau) = \sigma_N^2 \Phi_{v^{20}}(\tau)$$

$$\Phi_{S^{30}}(\tau) = 0$$

$$\Phi_{S^{11}}(\tau) = R_S(\tau) = R_v(\tau) + R_N(\tau)$$

$$\Phi_{S^{22}}(\tau) = \Phi_{v^{22}}(\tau) + \Phi_{N^{22}}(\tau) + 4R_v(\tau)R_N(\tau) + 2\sigma_v^2\sigma_N^2$$

$$\begin{aligned} \Phi_{S^{33}}(\tau) = & \Phi_{v^{33}}(\tau) + \Phi_{N^{33}}(\tau) + 6\Phi_{v^{13}}(\tau)\sigma_N^2 + 6\Phi_{N^{13}}(\tau)\sigma_v^2 + 9R_v(\tau)\Phi_{N^{22}}(\tau) \\ & + 9\Phi_{v^{22}}(\tau)R_N(\tau) + 18\Phi_{v^{12}}(\tau)\Phi_{N^{12}}(\tau) \end{aligned}$$

$$\Phi_{S^{12}}(\tau) = \Phi_{S^{21}}(\tau) = \Phi_{v^{12}}(\tau) + \Phi_{N^{12}}(\tau)$$

$$\Phi_{S^{13}}(\tau) = \Phi_{S^{31}}(\tau) = \Phi_{v^{13}}(\tau) + \Phi_{N^{13}}(\tau) + 3R_v(\tau)\sigma_N^2 + 3\sigma_v^2R_N(\tau)$$

$$\begin{aligned} \Phi_{S^{23}}(\tau) = & \Phi_{S^{32}}(\tau) = \Phi_{v^{23}}(\tau) + \Phi_{N^{23}}(\tau) + \Phi_{N^{30}}(\tau)\sigma_N^2 + 3\sigma_v^2\Phi_{N^{12}}(\tau) \\ & + 6R_v(\tau)\Phi_{N^{12}}(\tau) + 6\Phi_{v^{12}}(\tau)R_N(\tau) \end{aligned}$$

Table 5-1. Joint Moments of a Signal Plus Noise Mixed Process

$$E [v^n(t) v^{n'}(t + \tau)] = E [v^{n'}(t) v^n(t + \tau)]$$

$$E [N^n(t) N^{n'}(t + \tau)] = E [N^{n'}(t) N^n(t + \tau)]$$

5.3 JOINT MOMENTS OF $N(t)$

The joint moments of $N(t)$ are most readily obtained by successive differentiation of the joint characteristic function, as shown by Eq. (2-16).

$$\Phi_{x_{11'}}(\tau) = E \{x_1^{l_1} x_2^{l_2}\} = (-j)^{l_1+l_2} \left. \frac{\partial^{l_1+l_2} M(j\xi_1, j\xi_2)}{\partial \xi_1^{l_1} \partial \xi_2^{l_2}} \right|_{\xi_1=\xi_2=0} \quad (5-12)$$

The joint characteristic function is given by Eq. (2-14),

$$M(j\xi_1, j\xi_2) = \left\{ e^{(j\xi_1 x_1 + j\xi_2 x_2)} \right\} \quad (5-13)$$

If the pair of random variables x_1 and x_2 are sample functions of a Gaussian process, as $N(t)$ is, the joint characteristic function is given by ¹².

$$M(j\xi_1, j\xi_2) = e^{\left[j\xi_1 \bar{x}_1 + j\xi_2 \bar{x}_2 - \frac{1}{2} (\xi_1^2 \sigma_1^2 + \xi_2^2 \sigma_2^2 + 2\xi_1 \xi_2 \rho \sigma_1 \sigma_2) \right]} \quad (5-14)$$

where,

$$\sigma_1^2 = \overline{x_1^2} - \bar{x}_1^2 \quad \text{second central moment of } x_1$$

$$\sigma_2^2 = \overline{x_2^2} - \bar{x}_2^2 \quad \text{second central moment of } x_2$$

$$\rho = \frac{1}{\sigma_1 \sigma_2} E \left[(x_1 - \bar{x}_1) (x_2 - \bar{x}_2) \right] \quad \text{correlation coefficient}$$

Since $N(t)$ is described by a Gaussian process with zero mean and variance σ_N^2 , Eq. (5-14) reduces to,

$$M_N(j\xi_1, j\xi_2) = e^{-\frac{1}{2} (\xi_1^2 \sigma_N^2 + \xi_2^2 \sigma_N^2 + 2\xi_1 \xi_2 \rho_N \sigma_N^2)} \quad (5-15)$$

¹²Middleton, (REF. 6, p. 337).

$$\Phi_{N^{11'}}(\tau) = E \{ N^1(t) N^{1'}(t + \tau) \}$$

$$\Phi_{N^{10}}(\tau) = 0$$

$$\Phi_{N^{20}}(\tau) = \sigma_N^2$$

$$\Phi_{N^{30}}(\tau) = 0$$

$$\Phi_{N^{11}}(\tau) = R_N(\tau)$$

$$\Phi_{N^{22}}(\tau) = \sigma_N^4 + 2R_N^2(\tau)$$

$$\Phi_{N^{33}}(\tau) = 9\sigma_N^4 R_N(\tau) + 6R_N^3(\tau)$$

$$\Phi_{N^{12}}(\tau) = \Phi_{N^{21}}(\tau) = 0$$

$$\Phi_{N^{13}}(\tau) = \Phi_{N^{31}}(\tau) = 3\sigma_N^2 R_N(\tau)$$

$$\Phi_{N^{23}}(\tau) = \Phi_{N^{32}}(\tau) = 0$$

Table 5-2. The Joint Moments of $N(t)$

In addition, since $x_1 = N(t)$ and $x_2 = N(t + \tau)$,

$$\rho_N = \frac{1}{\sigma_N^2} E [N(t) N(t + \tau)] = \frac{R_N(\tau)}{\sigma_N^2} \quad (5-16)$$

Therefore, from Eq. (5-12),

$$\Phi_{N^{11'}}(\tau) = (-j)^{1+1'} \frac{\partial^{1+1'}}{\partial \epsilon_1^1 \partial \epsilon_2^{1'}} \left[e^{-\frac{1}{2}\sigma_N^2 \left(\epsilon_1^2 + \epsilon_2^2 + \frac{2R_N(\tau)}{\sigma_N^2} \epsilon_1 \epsilon_2 \right)} \right]_{\epsilon_1 = \epsilon_2 = 0}$$

or,

$$\Phi_{N^{11'}}(\tau) = (-j)^{1+1'} \frac{\partial^{1+1'}}{\partial \xi_1^1 \partial \xi_2^{1'}} \left[e^{\left(-\frac{1}{2} \sigma_N^2 \xi_1^2 - \frac{1}{2} \sigma_N^2 \xi_2^2 - R_N(\tau) \xi_1 \xi_2 \right)} \right]_{\xi_1 = \xi_2 = 0} \quad (5-17)$$

The moments, joint moments, and joint mixed moments of $N(t)$ are listed in Table 5-2 for 1 and 1' out to three.

5.4 OUTPUT AUTOCORRELATION FUNCTION FOR SIGNAL PLUS NOISE INPUT

Equations (5-9), (5-10), and (5-11), which express the statistical moments of $s(t)$, can now be written as,

$$\Phi_{S^{n0}}(\tau) = \sum_{l=0,2,4}^n \frac{[1 \cdot 3 \cdot 5 \dots (n-1)] n! \sigma_N^n}{\alpha! 1!} \Phi_{v^{n0}}(\tau) \quad (5-18)$$

$$\Phi_{S^{nn}}(\tau) =$$

$$\sum_{l=0}^n \sum_{l'=0}^n \frac{(n!)^2}{\alpha! \beta! 1! 1!} (-j)^{1+1'} \frac{\partial^{1+1'}}{\partial \xi_1^1 \partial \xi_2^{1'}} M_N(j \xi_1, j \xi_2) \Big|_{\xi_1 = \xi_2 = 0} \Phi_{v^{\alpha\beta}}(\tau)$$

$$\Phi_{S^{nn'}}(\tau) =$$

$$\sum_{l=0}^n \sum_{l'=0}^{n'} \frac{n! n!}{\alpha! \gamma! 1! 1!} (-j)^{1+1'} \frac{\partial^{1+1'}}{\partial \xi_1^1 \partial \xi_2^{1'}} M_N(j \xi_1, j \xi_2) \Big|_{\xi_1 = \xi_2 = 0} \Phi_{v^{\alpha\gamma}}(\tau)$$

The above equation express the joint moments of the input, $s(t)$, in terms of the joint moments of the signal, $v(t)$, in the presence of additive Gaussian noise with a characteristic function given by $M_N(j \xi_1, j \xi_2)$.

Insertion of the above equations into Eq. (4-5) gives the output autocorrelation function for the converter in terms of the joint moments of $v(t)$.

$$\begin{aligned}
R_0(\tau) = & A_{00}^2 + a_0 \cos p\omega_0 \tau + a_1 \Phi_{\nu a_0}(\tau) \\
& + a_2 \cos p\omega_0 \tau \Phi_{\nu a_0}(\tau) + a_3 \Phi_{\nu a\beta}(\tau) \\
& + a_4 \cos p\omega_0 \tau \Phi_{\nu a\beta}(\tau) + a_5 \Phi_{\nu a\delta}(\tau) \\
& + a_6 \cos p\omega_0 \tau \Phi_{\nu a\delta}(\tau)
\end{aligned} \tag{5-21}$$

The 'a' coefficients of Eq. (5-21) are given by,

$$\begin{aligned}
a_0 &= \frac{1}{2} \sum_{p=1}^{\infty} A_{0p}^2 \\
a_1 &= 2A_{00} \sum_{n=2,4,6,\dots}^{\infty} \sum_{l=0,2,4,\dots}^n \frac{A_{n0} [1 \cdot 3 \cdot 5 \dots (n-1)] n! \omega_N^n}{\alpha! 1!} \\
a_2 &= \sum_{n=2,4,6,\dots}^{\infty} \sum_{p=1}^{\infty} \sum_{l=0,2,4}^n \frac{A_{0n} A_{np} [1 \cdot 3 \cdot 5 \dots (n-1)] n! \omega_N^n}{\alpha! 1!} \\
a_3 &= \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{l=0}^n \sum_{l'=0}^n \frac{A_{n0}^2 (n!)^2}{\alpha! \beta! 1! 1'!} M_N \\
a_4 &= \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{l=0}^n \sum_{l'=0}^n \frac{1}{2} \frac{A_{np}^2 (n!)^2}{\alpha! \beta! 1! 1'!} M_N \\
a_5 &= \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \sum_{p=1}^{\infty} \sum_{l=0}^n \sum_{l'=0}^{n'} \frac{A_{n0} A_{n'0} n! n'!}{\alpha! \gamma! 1! 1'!} M_N \\
a_6 &= \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \sum_{p=1}^{\infty} \sum_{l=0}^n \sum_{l'=0}^{n'} \frac{A_{np} A_{n'p} n! n'!}{\alpha! \gamma! 1! 1'!} M_N
\end{aligned}$$

where,

$$M_N = (-j)^{l+l'} \frac{\partial^{l+l'}}{\partial \xi_1^l \partial \xi_2^{l'}} \left[M_N(j\xi_1, j\xi_2) \right] \Phi_{N^{ll'}}(\tau) \quad \xi_1 = \xi_2 = 0$$

The complete solution for $R_0(\tau)$ requires that the joint moments of $v(t)$ be specified. The following three sections will develop these moments for specific input signal voltage waveforms.

6. SINUSOIDAL SIGNAL INPUT

The first input waveform to be considered consists of a constant amplitude cosine wave of the form,

$$v(t) = A_0 \cos(\omega_c t + \phi) \quad (6-1)$$

where ϕ is uniformly distributed over the primary interval 0 to 2π .

The input shown above is a completely deterministic process, since the future behavior of the function at any time t is available once the phase ϕ has been specified.

In this section, the joint moments for the above input will be generated, and the output spectrum with and without noise present will be presented.

6.1 GENERATION OF JOINT MOMENTS

The n^{th} moment of $v(t)$ is given by,

$$\Phi_{v_{no}}(\tau) = E[v^n(t)] = A_0^n E[\cos^n \omega_c t]$$

where ϕ is set equal to zero for convenience.

The n^{th} power of the cosine function can be expressed as,

$$\cos^n \omega_c t = \left(\frac{1}{2}\right)^n [e^{j\omega_c t} + e^{-j\omega_c t}]^n$$

Introducing the binomial expansion,

$$\cos^n \omega_c t = \left(\frac{1}{2}\right)^n \sum_{y=0}^n \frac{n!}{(n-y)! y!} e^{j(n-2y)\omega_c t}$$

Re-arranging the terms to form cosines results in,

$$\cos^n \omega_c t = \left(\frac{1}{2}\right)^n \frac{n!}{\left(\frac{n}{2}\right)!^2} + \left(\frac{1}{2}\right)^{n-1} \sum_{y=0}^{\frac{n-2}{2}} \frac{n!}{(n-y)! y!} \cos(n-2y)\omega_c t, \quad (6-2)$$

for n even

$$= \left(\frac{1}{2}\right)^{n-1} \sum_{y=0}^{\frac{n-1}{2}} \frac{n!}{(n-y)! y!} \cos(n-2y)\omega_c t, \quad \text{for } \underline{n \text{ odd}} \quad (6-2) \text{ contd}$$

Since the expectation value of a cosine is zero, only the constant term in Eq. (6-2) will contribute to $\Phi_{v^{n0}}(\tau)$. Therefore,

$$\begin{aligned} \Phi_{v^{n0}}(\tau) &= \left(\frac{A_0}{2}\right)^n \frac{n!}{\left[\left(\frac{n}{2}\right)!\right]^2} && \text{for } \underline{n \text{ even}} \\ &= 0 && \text{for } \underline{n \text{ odd}} \end{aligned} \quad (6-3)$$

Similarly, Eq. (6-2) can be applied to the joint moments of $v(t)$, which are expressed as,

$$\Phi_{v^{nn}}(\tau) = E \left[A_0^{2n} \cos^n \omega_c t \cos^n \omega_c (t + \tau) \right] \quad (6-4)$$

Therefore,

For n even:

$$\begin{aligned} \Phi_{v^{nn}}(\tau) &= A_0^{2n} E \left\{ \left(\frac{1}{2}\right)^{2n} \frac{(n!)^2}{\left(\frac{n}{2}!\right)^4} \right\} \\ &+ A_0^{2n} E \left\{ \left(\frac{1}{2}\right)^{2(n-1)} \sum_{y=0}^{\frac{n-2}{2}} \sum_{y'=0}^{\frac{n-2}{2}} \frac{(n!)^2}{(n-y)! y! (n-y')! y'!} \cos(n-2y)\omega_c t \cos(n-2y')\omega_c (t + \tau) \right\} \end{aligned}$$

For n odd:

$$\begin{aligned} \Phi_{v^{nn}}(\tau) &= \\ &A_0^{2n} E \left\{ \left(\frac{1}{2}\right)^{2(n-1)} \sum_{y=0}^{\frac{n-1}{2}} \sum_{y'=0}^{\frac{n-1}{2}} \frac{(n!)^2}{(n-y)! y! (n-y')! y'!} \cos(n-2y)\omega_c t \cos(n-2y')\omega_c (t + \tau) \right\} \end{aligned} \quad (6-4)$$

All other terms will vanish, since they are expectation values of the product of two cosine functions of different frequencies, (see Eq. (2-25)). Replacing the terms above with their expectation values will give,

For n even:

$$\Phi_{v_{nn}}(\tau) = \left(\frac{A_0}{2}\right)^{2n} \frac{(n!)^2}{\left(\frac{n}{2}!\right)^4} + 2\left(\frac{A_0}{2}\right)^{2n} \sum_{y=0}^{\frac{n-2}{2}} \left[\frac{n!}{(n-y)! y!} \right]^2 \cos(n-2y)\omega_c \tau$$

For n odd:

$$\Phi_{v_{nn}}(\tau) = 2 \frac{A_0^{2n}}{2} \sum_{y=0}^{\frac{n-1}{2}} \left[\frac{n!}{(n-y)! y!} \right]^2 \cos(n-2y)\omega_c \tau \quad (6-5)$$

The joint mixed moments for $v(t)$ can be generated in the same manner. These moments are given by

$$\Phi_{v_{nn'}}(\tau) = E \left[A_0^{(n+n')} \cos^n \omega_c t \cos^{n'} \omega_c (t + \tau) \right]$$

or,

$$\Phi_{v_{nn'}}(\tau) = A_0^{(n+n')} E \left[\cos^n \omega_c t \cos^{n'} \omega_c (t + \tau) \right]$$

Applying Eq. (6-2) to the above, four sets of non-vanishing terms are generated.

For n and n' even:

$$\Phi_{v_{nn'}}(\tau) = A_0^{(n+n')} \frac{n! n!}{\left(\frac{n}{2}!\right)^2 \left(\frac{n'}{2}!\right)^2} \quad (6-6)$$

$$+ \frac{1}{2} \sum_{y=0}^{\frac{n-2}{2}} \sum_{y'=0}^{\frac{n'-2}{2}} \frac{n! n!}{(n-y)! y! (n'-y')! y'!} \cos(n-2y)\omega_c \tau$$

$$v(t) = A_0 \cos \omega_c t$$

$$\Phi_{v^n v^m}(\tau) = E \{ v^n(t) v^m(t + \tau) \}$$

$$\Phi_{v10}(\tau) = 0$$

$$\Phi_{v20}(\tau) = \sigma_v^2 = \frac{1}{2} A_0^2$$

$$\Phi_{v30}(\tau) = 0$$

$$\Phi_{v11}(\tau) = R_v(\tau) = \frac{1}{2} A_0^2 \cos \omega_c \tau$$

$$\Phi_{v22}(\tau) = \frac{1}{2} \sigma_v^4 + R_v^2(\tau) = \frac{1}{4} A_0^4 + \frac{1}{8} A_0^4 \cos 2\omega_c \tau$$

$$\Phi_{v33}(\tau) = \frac{3}{2} \sigma_v^4 R_v(\tau) + R_v^3(\tau) = \frac{9}{32} A_0^6 \cos \omega_c \tau + \frac{1}{32} A_0^6 \cos 3\omega_c \tau$$

$$\Phi_{v12}(\tau) = \Phi_{v21}(\tau) = 0$$

$$\Phi_{v13}(\tau) = \frac{3}{2} \sigma_v^2 R_v(\tau) = \frac{3}{8} A_0^4 \cos \omega_c \tau$$

$$\Phi_{v31}(\tau) = \frac{3}{2} \sigma_v^2 R_v(\tau) = \frac{3}{8} A_0^4 \cos \omega_c \tau$$

$$\Phi_{v23}(\tau) = \Phi_{v32}(\tau) = 0$$

Table 6-1. Joint Moments of Cosine Wave

For n and n' odd, or n even and n' odd, or n odd and n' even:

$$\Phi_{v_{nn'}}(\tau) = \left(\frac{1}{2}\right)^{(n+n'-1)} \sum_{y=0}^Y \sum_{y'=0}^{Y'} \frac{n! n'}{(n-y)! y! (n'-y')! y'!} \cos(n-2y) \cos(n'-2y') \quad (6-6)$$

where, (b)

$$Y = n - 2/2 \text{ (for } n \text{ even)} = n - 1/2 \text{ (for } n \text{ odd)}$$

$$Y' = n' - 2/2 \text{ (for } n' \text{ even)} = n' - 1/2 \text{ (for } n' \text{ odd)}$$

Only terms where $(n - n') = 2(y - y')$ can exist in Eq. (6-6) (b), because all others are cross-correlations of cosine functions of different frequency.

Table 6-1 lists the moments of $v(t)$ for n and n' out to three. All joint mixed moments whose sum $n + n'$ is odd do not exist, hence only those moments for which $n + n'$ is even need be considered in Eq.'s (6-5) and (6-6).

6.2 OUTPUT SPECTRUM FOR COSINE INPUT

The output autocorrelation function for the frequency converter with a cosine input can be found by inserting the joint moments of Eq.'s (6-3), (6-5), and (6-6) into Eq. (4-5).

Eq. (4-5) is shown below for n and n' out to three.

$$\begin{aligned} R_0(\tau) = & A_{00}^2 + \frac{1}{2} \left[A_{01}^2 \cos \omega_0 \tau + A_{02}^2 \cos 2\omega_0 \tau + A_{03}^2 \cos 3\omega_0 \tau \right] \\ & + 2A_{00} \left[A_{10} \Phi_{S10}(\tau) + A_{20} \Phi_{S20}(\tau) + A_{30} \Phi_{S30}(\tau) \right] \\ & + \sum_{n=1}^3 \sum_{p=1}^3 A_{0p} A_{np} \cos p\omega_0 \tau \Phi_{S_{n0}}(\tau) \\ & + A_{10}^2 \Phi_{S11}(\tau) + A_{20}^2 \Phi_{S22}(\tau) + A_{30}^2 \Phi_{S33}(\tau) \end{aligned} \quad (6-7)$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{n=1}^3 \sum_{p=1}^3 A_{np}^2 \cos p \omega_0 \tau \Phi_{S^{nn}}(\tau) \\
& + 2 A_{10} A_{20} \Phi_{S^{12}}(\tau) + 2 A_{10} A_{30} \Phi_{S^{13}}(\tau) + 2 A_{20} A_{10} \Phi_{S^{21}}(\tau) \quad (6-7) \\
& + 2 A_{20} A_{30} \Phi_{S^{23}}(\tau) + 2 A_{10} A_{30} \Phi_{S^{31}}(\tau) + 2 A_{20} A_{30} \Phi_{S^{32}}(\tau) \quad \text{contd}
\end{aligned}$$

Utilizing the values of Table 6-1, Eq. (6-7) will be,

$$\begin{aligned}
R_0(\tau) = & A_{00}^2 + 2 A_{00} A_{20} \sigma_v^2 + \frac{1}{2} A_{20}^2 \sigma_v^4 \\
& + \left[\frac{1}{2} A_{0p}^2 + A_{0p} A_{2p} \sigma_v^2 + \frac{1}{4} A_{2p}^2 \sigma_v^4 \right] \cos p \omega_0 \tau \\
& + \left[A_{10}^2 + \frac{3}{2} A_{30}^2 \sigma_v^4 + 6 A_{10} A_{30} \sigma_v^2 \right] R_v(\tau) \quad (6-8) \\
& + A_{20}^2 R_v^2(\tau) + \frac{1}{2} A_{2p}^2 \cos p \omega_0 \tau R_v^2(\tau) \\
& + A_{30}^2 R_v^3(\tau) + \frac{1}{2} A_{3p}^2 \cos p \omega_0 \tau R_v^3(\tau) \\
& + \left[\frac{1}{2} A_{1p}^2 + \frac{3}{4} A_{3p}^2 \sigma_v^4 + 3 A_{1p} A_{3p} \sigma_v^2 \right] \cos p \omega_0 \tau R_v(\tau)
\end{aligned}$$

where p is summed from 1 to 3.

The power spectral density at the converter output is obtained from the Fourier transform of each term in Eq. (6-8). The transforms of the product terms, as shown in Section 2.5, are determined from the convolution,

$$F \left[R_1(\tau) R_2(\tau) \right] = S_1(f) * S_2(f)$$

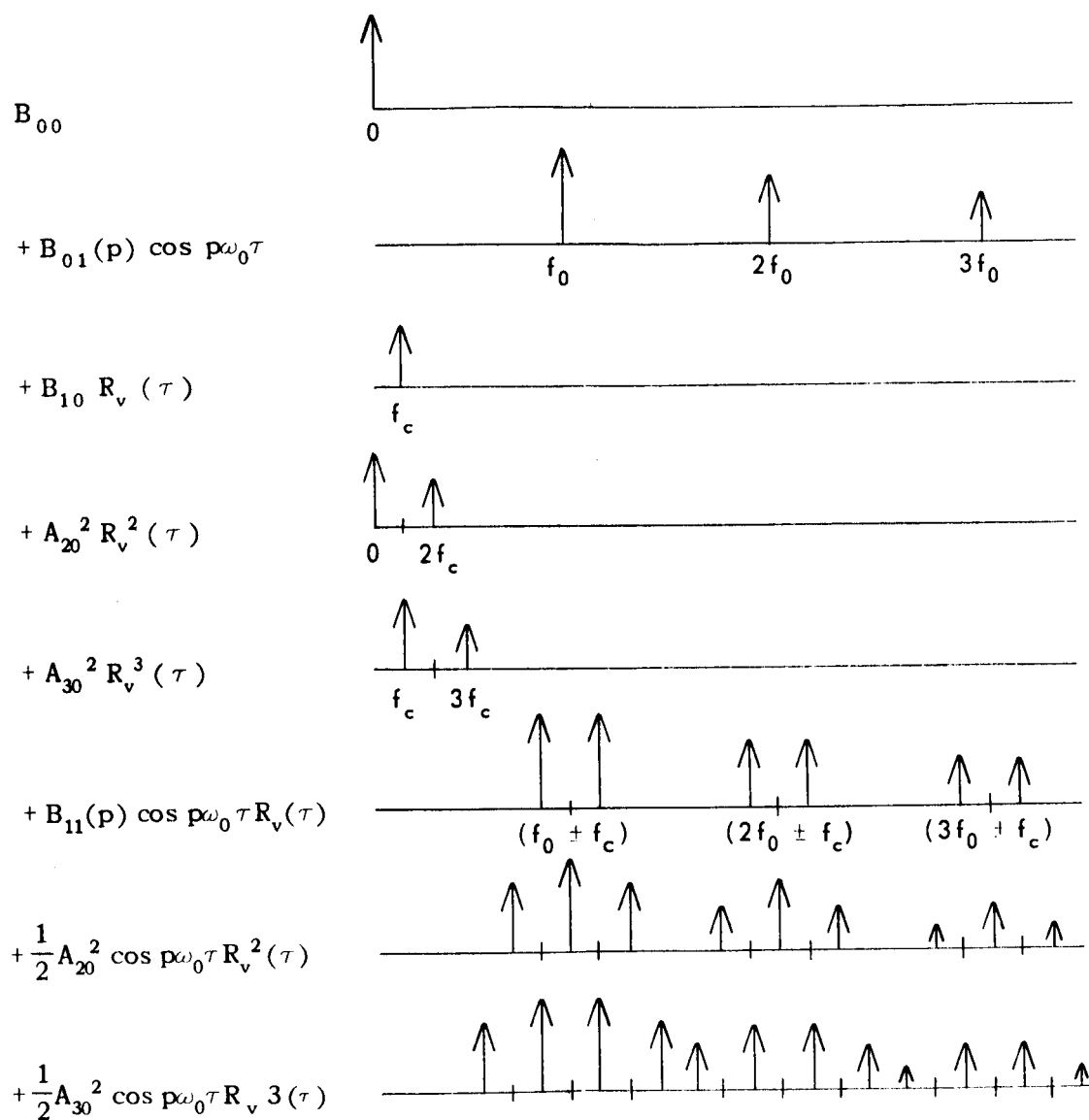
The contributions of each term of $R_0(\tau)$ to the output spectrum are shown in Figure 6-1, where the constant coefficients are given by,

$$\begin{aligned}
 B_{00} &= \left[A_{00}^2 + 2 A_{00} A_{20} \sigma_v^2 + \frac{1}{2} A_{20}^2 \sigma_v^2 \right] \\
 B_{01}(p) &= \left[\frac{1}{2} A_{0p}^2 + A_{0p} A_{2p} \sigma_v^2 + \frac{1}{4} A_{2p}^2 \sigma_v^4 \right] \\
 B_{10} &= \left[A_{10}^2 + \frac{3}{2} A_{30}^2 \sigma_v^4 + 6 A_{10} A_{30} \sigma_v^2 \right] \\
 B_{11}(p) &= \left[\frac{1}{2} A_{1p}^2 + \frac{3}{4} A_{3p}^2 \sigma_v^4 + 3 A_{1p} A_{3p} \sigma_v^2 \right]
 \end{aligned} \tag{6-9}$$

The complete output spectrum expressed in terms of the 'A' coefficients of the diode can now be found by utilizing Figures (2-2) and (2-3) to obtain the spectral component amplitudes. Then,

$$\begin{aligned}
 S_0(f) &= \left[B_{00} + \frac{1}{8} A_0^4 A_{20}^2 \right] \delta(f \pm 0) \\
 &+ \left[\frac{1}{4} A_0^2 B_{10} + \frac{3}{64} A_0^6 A_{30}^2 \right] \delta(f \pm f_c) \\
 &+ \left[\frac{1}{16} A_0^4 A_{20}^2 \right] \delta(f \pm 2f_c) \\
 &+ \left[\frac{1}{128} A_0^6 A_{30}^2 \right] \delta(f \pm 3f_c) \\
 &+ \left[\frac{1}{2} B_{01}(p) + \frac{1}{32} A_c^4 A_{2p}^2 \right] \delta(f \pm p f_0)
 \end{aligned} \tag{6-10}$$

$$R_0(\tau) =$$



Note: Amplitudes of spectral components not drawn to scale.

Figure 6-1. Spectral Contributions of the Components of $R_0(\tau)$ for a Single Carrier Input

$$\begin{aligned}
& + \left[\frac{1}{8} A_c^2 B_{11}(p) + \frac{3}{256} A_c^2 A_{3p}^2 \right] \delta \left[f \pm (p f_0 + f_c) \right] \\
& + \left[\frac{1}{64} A_c^4 A_{2p}^2 \right] \delta \left[f \pm (p f_0 \pm 2 f_c) \right] \\
& + \left[\frac{1}{512} A_c^6 A_{3p}^2 \right] \delta \left[f \pm (p f_0 \pm 3 f_c) \right]
\end{aligned} \tag{6-10}$$

contd

The relative magnitudes of the spectral components of $S_0(f)$ can be observed by utilizing the 'A' coefficients listed in Appendix II, Figure II-4, for the 1N53C diode. The output components of $S_0(f)$ for this diode are displayed in Figure 6-2, where E_0 is 0.25v and A_0 is 0.01v. The magnitude of the spectral components is plotted in db, with the zero db reference set at the f_c level at the input to the converter.

The conversion loss of the converter for fundamental mixing is 7.2 db. The output filtering requirements will be primarily dictated by the local oscillator spike at f_0 , which is f_c cps away from the desired output frequency and 23 db above the desired output frequency level.

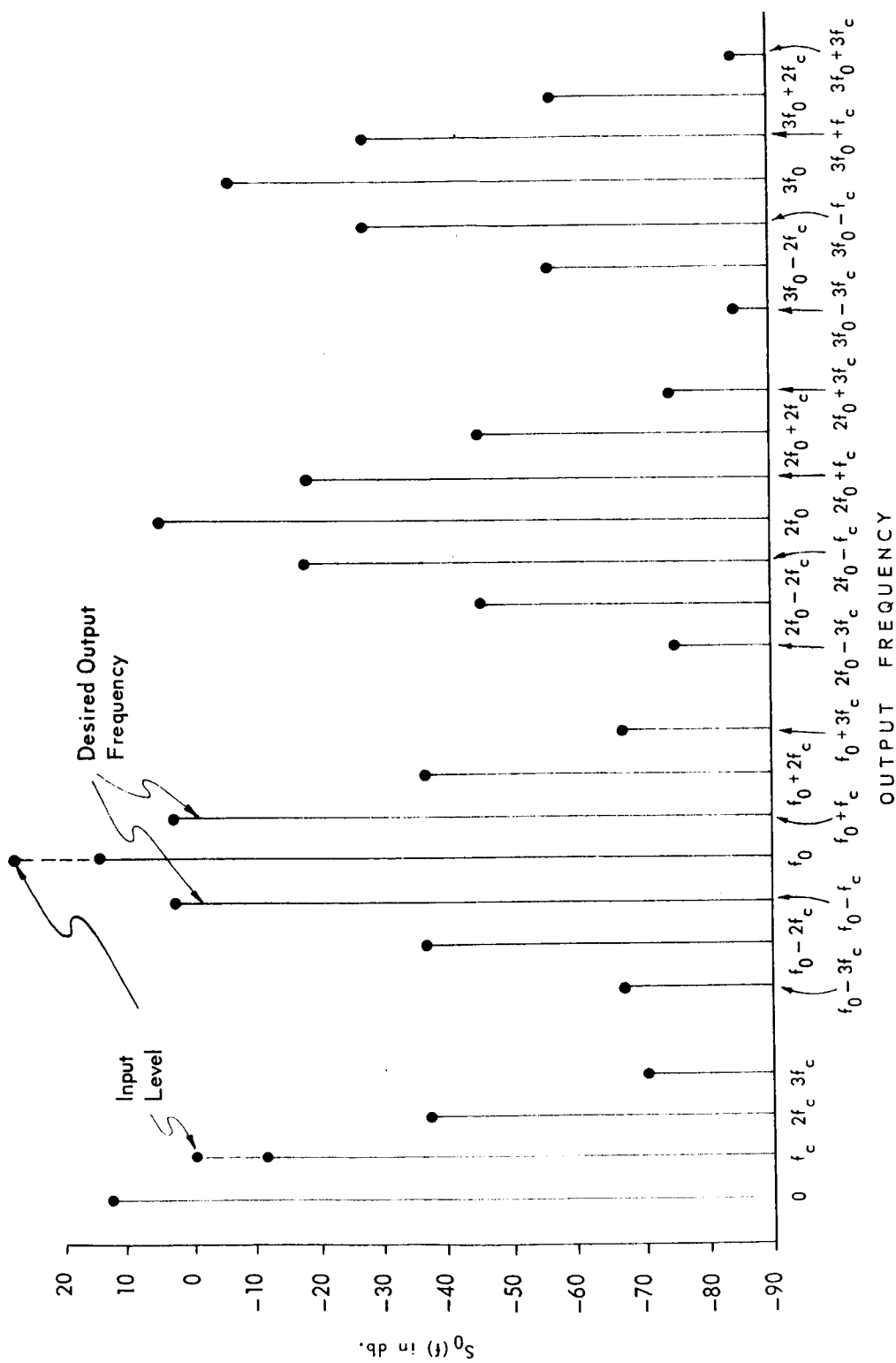


Figure 6-2. Output Power Spectrum for IN53C Diode Converter with a Single Carrier Input

6.3 OUTPUT SPECTRUM FOR COSINE PLUS NOISE INPUT

The output autocorrelation function for the frequency converter with a signal plus noise input is shown as Eq. (5-21). Insertion of the moments of Eq.'s (6-3), (6-5), and (6-6) into that equation results in the output autocorrelation function for a cosine plus narrowband noise input.

The non-vanishing moments of the cosine plus noise input for n and n' out to three are listed in Table 6-2 as a function of σ_v^2 , $R_v(\tau)$, σ_N^2 , and $R_N(\tau)$. These moments were obtained by applying the joint moments of Table's 6-1 and 5-2 to the expressions listed in Table 5-1 for the signal plus noise condition.

The complete output autocorrelation function for the converter for n and n' out to three is shown below:

$$\begin{aligned}
 R_0(\tau) = & A_{00}^2 + 2A_{00}A_{20}\sigma_N^2\sigma_v^2 + 2A_{20}^2\sigma_v^2\sigma_N^2 + \frac{1}{2}A_{20}^2\sigma_v^4 + A_{20}^2\sigma_N^4 \\
 & + \left[\frac{1}{2}A_{0p}^2 + A_{0p}A_{2p}\sigma_v^2\sigma_N^2 + \frac{1}{4}A_{2p}^2\sigma_v^4 \right. \\
 & \left. + \frac{1}{2}A_{2p}^2\sigma_N^4 + A_{2p}^2\sigma_v^2\sigma_N^2 \right] \cos p\omega_0\tau \\
 & + \left[A_{10}^2 + \frac{3}{2}A_{30}^2\sigma_v^4 + 6A_{10}A_{30}\sigma_v^2 \right. \\
 & \left. + 12A_{10}A_{30}\sigma_N^2 + 9A_{30}^2\sigma_N^2(\sigma_N^2 + \sigma_v^2) \right] R_v(\tau) \\
 & + A_{20}^2 R_v^2(\tau) + \frac{1}{2}A_{2p}^2 \cos p\omega_0\tau R_v^2(\tau) \\
 & + A_{30}^2 R_v^3(\tau) + \frac{1}{2}A_{3p}^2 \cos p\omega_0\tau R_v^3(\tau)
 \end{aligned} \tag{6-11}$$

$$\begin{aligned}
& + \left[\frac{1}{2} A_{1p}^2 + \frac{3}{4} A_{3p}^2 \sigma_v^4 + 3 A_{1p} A_{3p} \sigma_v^2 \right. \\
& \quad \left. + 6 \sigma_N^2 A_{1p} A_{3p} + \frac{9}{2} A_{3p}^2 \sigma_N^2 (\sigma_N^2 + \sigma_v^2) \right] \cos p \omega_0 \tau R_v(\tau) \\
& + \left[A_{10}^2 + 4 A_{10} A_{30} (3 \sigma_v^2 + 3 \sigma_N^2) + A_{30}^2 \left(\frac{9}{2} \sigma_v^4 + 18 \sigma_v^2 \sigma_N^2 + 9 \sigma_N^4 \right) \right] R_N(\tau) \\
& + 2 A_{20}^2 R_N^2(\tau) + A_{2p}^2 \cos p \omega_0 \tau R_N^2(\tau) \\
& + 6 A_{30}^2 R_N^3(\tau) + 3 A_{3p}^2 \cos p \omega_0 \tau R_N^3(\tau) \\
& + \left[\frac{1}{2} A_{1p}^2 + 6 A_{1p} A_{3p} (\sigma_v^2 + \sigma_N^2) \right. \\
& \quad \left. + \frac{1}{2} A_{3p}^2 \left(\frac{9}{2} \sigma_v^4 + 18 \sigma_v^2 \sigma_N^2 + 9 \sigma_N^4 \right) \right] \cos p \omega_0 \tau R_N(\tau) \\
& + [4 A_{20}^2 + 2 A_{2p}^2 \cos p \omega_0 \tau] R_v(\tau) R_N(\tau) \\
& + \left[9 A_{30}^2 + \frac{9}{2} A_{3p}^2 \cos p \omega_0 \tau \right] R_v^2(\tau) R_N(\tau) \\
& + [18 A_{30}^2 + 9 A_{3p}^2 \cos p \omega_0 \tau] R_v(\tau) R_N^2(\tau)
\end{aligned}$$

(6-11)

cont'd

A comparison of the above equation with Eq. (6-8) shows that the presence of the narrowband noise has altered the amplitude of the first order terms, (i.e., dc, $\cos \omega_0 \tau$, $R_v(\tau)$, and $\cos p \omega_0 \tau R_v(\tau)$), but has left unchanged the second and

third order signal terms, $R_v^2(\tau)$, $R_v^3(\tau)$, etc.). In addition, a complete set of noise-only terms has been generated which are similar in form to the signal terms. Mixed third order terms, involving both $R_v(\tau)$ and $R_N(\tau)$ are also present.

The output power spectrum for the converter with a cosine plus noise input can be obtained from the Fourier transform of each term in Eq. (6-11). The positive frequency spectral contributions of each term of $R_0(\tau)$ are displayed in Figure 6-3. The coefficients are given by,

$$B_{00'} = \left[A_{00}^2 + 2A_{00}A_{20}\sigma_N^2\sigma_v^2 + 2A_{20}^2\sigma_v^2\sigma_N^2 + \frac{1}{2}A_{20}^2\sigma_v^4 + A_{20}^2\sigma_N^4 \right]$$

$$B_{01'}(p) = \left[\frac{1}{2}A_{0p}^2 + A_{0p}A_{2p}\sigma_v^2\sigma_N^2 + \frac{1}{4}A_{2p}^2\sigma_v^4 + \frac{1}{2}A_{2p}^2\sigma_N^4 + A_{2p}^2\sigma_v^2\sigma_N^2 \right]$$

$$B_{10'} = \left[A_{10}^2 + \frac{3}{2}A_{30}^2\sigma_v^4 + 6A_{10}A_{30}\sigma_v^2 + 12A_{10}A_{30}\sigma_N^2 + 9A_{30}^2\sigma_N^2(\sigma_N^2 + \sigma_v^2) \right]$$

$$B_{11'}(p) = \left[\frac{1}{2}A_{1p}^2 + \frac{3}{4}A_{3p}^2\sigma_v^4 + A_{1p}A_{3p}(3\sigma_v^2 + 6\sigma_N^2) + \frac{9}{2}A_{3p}^2\sigma_N^2(\sigma_N^2 + \sigma_v^2) \right]$$

$$\Gamma_{10} = \left[A_{10}^2 + 12A_{10}A_{30}(\sigma_v^2 + \sigma_N^2) + A_{30}^2\left(\frac{9}{2}\sigma_v^4 + 18\sigma_v^2\sigma_N^2 + 9\sigma_N^4\right) \right]$$

$$\Gamma_{11}(p) = \left[\frac{1}{2}A_{1p}^2 + 6A_{1p}A_{3p}(\sigma_v^2 + \sigma_N^2) + \frac{1}{2}A_{3p}^2\left(\frac{9}{2}\sigma_v^4 + 18\sigma_v^2\sigma_N^2 + 9\sigma_N^4\right) \right]$$

The Fourier transform of the terms of $R_0(\tau)$ that involve the product of $R_N(\tau)$ with itself or with $\cos p\omega_0\tau$ can be expressed as $S_N(f)$ displaced by $\pm pf_0$, i.e., $S_N(f \pm pf_0)$. This can be shown by the following:

$$\begin{aligned} F[\cos p\omega_0\tau R_N(\tau)] &= F[\cos p\omega_0\tau] * S_N(f) \\ &= \frac{1}{2}\delta(f \pm pf_0) * S_N(f) \\ &= \frac{1}{2}S_N(f \pm pf_0) \end{aligned}$$

$$S(t) = A_0 \cos(\omega_c t + \phi) + N(t)$$

$$\Phi_{S^{nn}}(\tau) = E \{ S^n(t) S^{n'}(t + \tau) \}$$

$$\Phi_{S^{20}}(\tau) = \sigma_N^2 \sigma_v^2$$

$$\Phi_{S^{11}}(\tau) = R_v(\tau) + R_N(\tau)$$

$$\Phi_{S^{22}}(\tau) = \left[\frac{1}{2} \sigma_v^4 + \sigma_N^4 + 2 \sigma_v^2 \sigma_N^2 \right] + 4 R_v(\tau) R_N(\tau) + R_v^2(\tau) + 2 R_N^2(\tau)$$

$$\begin{aligned} \Phi_{S^{33}}(\tau) = & R_v^3(\tau) + 6 R_N^3(\tau) + 9 R_v^2(\tau) R_N(\tau) + 18 R_v(\tau) R_N^2(\tau) \\ & + \left[\frac{3}{2} \sigma_v^4 + 9 \sigma_v^2 \sigma_N^2 + 9 \sigma_N^4 \right] R_v(\tau) \\ & + \left[\frac{9}{2} \sigma_v^4 + 18 \sigma_v^2 \sigma_N^2 + 9 \sigma_N^4 \right] R_N(\tau) \end{aligned}$$

$$\Phi_{S^{13}}(\tau) = \Phi_{S^{31}}(\tau) = \left[\frac{3}{2} \sigma_v^2 + 3 \sigma_N^2 \right] R_v(\tau) + \left[3 \sigma_v^2 + 3 \sigma_N^2 \right] R_N(\tau)$$

$$\sigma_v^2 = \frac{1}{2} A_c^2$$

$$R_v(\tau) = \frac{1}{2} A_c^2 \cos \omega_c \tau$$

$$\sigma_N^2 = 2 a_0 b$$

$$R_N(\tau) = 2 a_0 b \frac{\sin 2\pi \frac{b}{2} \tau}{2\pi \frac{b}{2} \tau} \cos \omega_c \tau$$

Table 6-2. Joint Moments of Cosine Plus Narrowband Noise

$$+ 9A_{3p}^2 \cos p\omega_0 \tau R_v(\tau) R_N^2(\tau)$$



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Similarly,

$$F [\cos p \omega_0 \tau R_N^2(\tau)] = \frac{1}{2} \delta (f \pm p f_0) * S_N(f) * S_N(f)$$

$$= \frac{1}{2} S_N(f \pm p f_0) * S_N(f \pm p f_0)$$

$$F [\cos p \omega_0 \tau R_N^3(\tau)] = \frac{1}{2} S_N(f \pm p f_0) * S_N(f \pm p f_0)$$

$$F [R_v(\tau) R_N(\tau)] = \frac{1}{4} A_c^2 \delta (f \pm f_c) * S_N(f)$$

$$= \frac{1}{4} A_c^2 S_N(f \pm f_c)$$

$$F [R_v^2(\tau) R_N(\tau)] = \frac{1}{2} A_c^2 S_N(f) + \frac{1}{4} A_c^2 S_N(f \pm 2 f_c)$$

$$F [R_v(\tau) R_N^2(\tau)] = \frac{1}{4} A_c^2 S_N(f \pm f_c) * S_N(f \pm f_c)$$

$$F [\cos p \omega_0 \tau R_v(\tau) R_N(\tau)] = \frac{1}{8} A_c^2 S_N(f \pm p f_0 \pm f_c)$$

$$F [\cos p \omega_0 \tau R_v^2(\tau) R_N(\tau)] = \frac{1}{4} A_c^2 S_N(f \pm p f_0) + \frac{1}{8} A_c^2 S_N(f \pm p f_0 \pm 2 f_c)$$

$$F [\cos p \omega_0 \tau R_v(\tau) R_N^2(\tau)] = \frac{1}{8} A_c^2 S_N(f \pm p f_0 \pm f_c) * S_N(f \pm p f_0 \pm f_c)$$

The amplitudes and form of the above spectra can be obtained from the figures of Section 2.5. All of the resulting spectra above involve $S_N(f)$ or a self-convolution of $S_N(f)$ centered about a particular frequency, therefore the results of Section 2.5 are applicable to the solution of $S_0(f)$.

The complete output spectrum for the frequency converter, expressed in terms of the A_{np} coefficients of the diode, can now be found by utilizing the above convolutions in Eq. (6-11). The resulting spectral density is,

$$\begin{aligned}
 S_0(f) = & \left[B_{00'} + \frac{1}{8} A_0^4 A_{20}^2 \right] \delta(f - 0) \\
 & + \left[\frac{1}{4} A_{20}^2 B_{10'} + \frac{3}{64} A_0^6 A_{30}^2 \right] \delta(f \pm f_c) \\
 & + \left[\frac{1}{16} A_0^4 A_{20}^2 \right] \delta(f \pm 2f_c) \\
 & + \left[\frac{1}{128} A_0^6 A_{30}^2 \right] \delta(f \pm 3f_c) \\
 & + \left[\frac{1}{2} B_{01'}(p) + \frac{1}{32} A_c^4 A_{2p}^2 \right] \delta(f \pm pf_0) \\
 & + \left[\frac{1}{8} A_c^2 B_{11'}(p) + \frac{3}{256} A_c^6 A_{3p}^2 \right] \delta(f \pm pf_0 \pm f_c) \\
 & + \left[\frac{1}{64} A_c^4 A_{2p}^2 \right] \delta(f \pm pf_0 \pm 2f_c) \\
 & + \left[\frac{1}{512} A_c^6 A_{3p}^2 \right] \delta(f \pm pf_0 \pm 3f_c)
 \end{aligned} \tag{6-13}$$

$$+ \left[\Gamma_{10} + \frac{9}{2} A_0^2 A_{30}^2 \right] S_N(f)$$

$$+ 2 A_{20}^2 S_N(f) * S_N(f)$$

$$+ 6 A_{30}^2 S_N(f) *^3 S_N(f)$$

$$+ \left[\frac{1}{2} \Gamma_{11}(p) + \frac{9}{8} A_c^2 A_{3p}^2 \right] S_N(f \pm p f_0)$$

$$+ \frac{1}{2} A_{2p}^2 S_N(f \pm p f_0) * S_N(f \pm p f_0)$$

$$+ \frac{1}{2} A_{3p}^2 S_N(f \pm p f_0) *^3 S_N(f \pm p f_0)$$

(6-13)
contd

$$+ A_{20}^2 A_c^2 S_N(f \pm f_c)$$

$$+ \frac{9}{4} A_c^2 A_{30}^2 S_N(f \pm 2 f_c)$$

$$+ \frac{9}{2} A_c^2 A_{30}^2 S_N(f \pm f_c) * S_N(f \pm f_c)$$

$$+ \frac{1}{4} A_c^2 A_{2p}^2 S_N(f \pm p f_0 \pm f_c)$$

$$+ \frac{9}{16} A_c^2 A_{3p}^2 S_N(f \pm p f_0 \pm 2 f_c)$$

$$+ \frac{9}{8} A_c^2 A_{3p}^2 S_N (f \pm p f_0 \pm f_c) * S_N (f \pm p f_0 \pm f_c) \quad (6-13) \\ \text{contd}$$

6.3.1 OUTPUT SIGNAL TO NOISE RATIO

The terms of $R_0(\tau)$ that will contribute noise power near one of the desired output frequencies, say at $f_0 - f_c$, can be seen from Figure 6-3 to be,

$$\Gamma_{11}(1) \cos \omega_0 \tau R_N(\tau)$$

$$A_{31}^2 \cos \omega_0 \tau R_N^3(\tau)$$

$$\frac{9}{2} A_{31}^2 \cos \omega_0 \tau R_v^2(\tau) R_N(\tau)$$

$$9 A_{31}^2 \cos \omega_0 \tau R_v(\tau) R_N^2(\tau)$$

The mean square value of the noise power near $f_0 - f_c$, N_0 , is the sum of the mean square values of the power that portion of the spectrum of each term above which falls near $f_0 - f_c$. That is,

$$N_0 = \frac{1}{4} \left[\frac{1}{2} \Gamma_{11}(1) \sigma_N^2 \right] + \frac{3}{16} \left[\frac{1}{2} A_{31}^2 \sigma_N^6 \right] \\ + \frac{1}{6} \left[\frac{9}{4} A_{31}^2 \sigma_v^4 \sigma_N^2 \right] + \frac{3}{16} \left[\frac{9}{2} A_{31}^2 \sigma_v^2 \sigma_N^4 \right]$$

where each bracketed value above corresponds to the total power of the contributing term and the multiplying fraction is that part of the spectrum which is near $f_0 - f_c$. Therefore

$$N_0 = \frac{1}{8} \sigma_N^2 \left[\Gamma_{11}(1) + \frac{3}{4} A_{31}^2 (\sigma_N^4 + 9 \sigma_v^2 \sigma_N^2 + \sigma_v^4) \right]$$

The signal power at $f_0 - f_c$, S_0 , can be found in a similar manner,

$$\begin{aligned} S_0 &= \frac{1}{4} \left[\frac{1}{2} B_{11'}(p) \sigma_v^2 \right] + \frac{1}{6} \left[\frac{1}{4} A_{31}^2 \sigma_v^6 \right] \\ &= \frac{1}{8} \sigma_v^2 \left[B_{11'}(1) + \frac{1}{3} A_{31}^2 \sigma_v^4 \right] \end{aligned}$$

The signal to noise power ratio at $f_0 - f_c$ is therefore

$$\left(\frac{S_0}{N_0} \right) = \frac{\sigma_v^2}{\sigma_N^2} \frac{\left[B_{11'}(1) + \frac{1}{3} A_{31}^2 \sigma_v^4 \right]}{\left[\Gamma_{11}(1) + \frac{3}{4} A_{31}^2 (\sigma_N^4 + 9 \sigma_v^2 \sigma_N^2 + \sigma_v^4) \right]} \quad (6-15)$$

Since the input signal to noise power ratio is given by

$$\left(\frac{S_i}{N_i} \right) = \frac{\sigma_v^2}{\sigma_N^2}$$

Therefore

$$\left(\frac{S_0}{N_0} \right) = \frac{\left[B_{11'}(1) + \frac{1}{3} A_{31}^2 \sigma_v^4 \right]}{\left[\Gamma_{11}(1) + \frac{3}{4} A_{31}^2 (\sigma_N^4 + 9 \sigma_v^2 \sigma_N^2 + \sigma_v^4) \right]} \left(\frac{S_i}{N_i} \right) \quad (6-16)$$

The above equation relates the output signal to noise ratio of the converter to the input signal to noise ratio, in terms of the diode constants and the mean square values of the input signal and noise.

7. AMPLITUDE MODULATED SIGNAL INPUT

This section will consider a converter input of a cosine carrier with a modulating envelope of the form

$$v(t)_{AM} = A_0 [1 + a A(t)] \cos(\omega_c t + \psi_0) \quad (7-1)$$

where: a is the modulation index, and

$A(t)$ is the modulating voltage, with $|aA(t)| < 1$, so that no overmodulation occurs.

The modulating voltage, $A(t)$, is assumed to possess a rectangular Gaussian spectrum centered at zero frequency and obeying the conditions listed in Section 5.1. Such a spectrum is a reasonable model to assume for the representation of many types of complex waveform spectra such as those of voice, television, frequency division multiplex telephony and other communication signals¹³.

7.1 AUTOCORRELATION FUNCTION OF AM WAVE

The autocorrelation function of $v(t)_{AM}$ is obtained from Eq. (2-30) as

$$R_v(\tau)_{AM} = \frac{A_0^2}{2} R_v(\tau) \cos \omega_c \tau \quad (7-2)$$

where $R_v(\tau)$ is the autocorrelation function of $V(t) = [1 + aA(t)]$.

The power spectral density is then

$$S_v(f)_{AM} = \frac{A_0^2}{4} S_v(f) * \delta(f \pm f_c) \quad (7-3)$$

$$= \frac{A_0^2}{4} S_v(f \pm f_c)$$

where $S_v(f)$ is the power spectral density of $V(t)$.

¹³Abramson, (REF. 1, p. 407), and Stewart, (REF. 13, p. 1539).

For

$$S_A(f) = a_0 \quad ; \quad -\frac{b}{2} < f < \frac{b}{2}$$

$$= 0 \quad ; \quad |f| > \frac{b}{2}$$

then

$$S_v(f) = \delta(f - 0) + a^2 S_A(f) \quad (7-4)$$

and

$$\begin{aligned} S_v(f)_{AM} &= \frac{A_0^2}{4} \left[\delta(f \pm f_c) + a^2 S_A(f \pm f_c) \right] \\ &= \frac{A_0^2}{4} \delta(f \pm f_c) + a^2 \frac{A_0^2}{4} S_A(f \pm f_c) \end{aligned} \quad (7-5)$$

This spectrum is identical in form to that of a cosine at f_c plus narrowband noise, $N(t)$, centered at $+f_c$ and $-f_c$, i.e., $v(t)_{AM}$ can be expressed as

$$v(t)_{AM} = A_0 \cos(\omega_c t + \phi) + a \frac{A_0}{2} N(t) \quad (7-6)$$

7.2 CONVERTER OUTPUT SPECTRUM FOR AM INPUT

The spectrum of $v(t)_{AM}$, as given by Eq. (7-6), is precisely the input spectrum considered in Section 6.3, except for the multiplying factor on $N(t)$. The results of that section may therefore be utilized to obtain the output autocorrelation function and power spectral density of the converter.

The joint moments of $v(t)_{AM}$ will be given by Table 6-2 with the following substitutions,

$$\sigma_N^2 = \frac{1}{2} a^2 a_0 b A_0^2 \quad (7-7)$$

$$R_N(\tau) = \frac{1}{2} a^2 a_0 b A_0^2 \frac{\sin 2\pi \frac{b}{2} \tau}{2\pi \frac{b}{2} \tau} \cos \omega_c \tau \quad (7-8)$$

With the above substitutions, the converter output autocorrelation function for a carrier amplitude-modulated by narrowband Gaussian noise input is given by Eq. (6-11). The complete output power spectrum for the frequency converter is given by Eq. (6-13), with $S_N(f)$ replaced by the modulating voltage of the AM wave, $S_A(f)$.

8. ANGLE MODULATED SIGNAL INPUT

In this section the input to the converter will be taken as a cosine wave which is phase or frequency modulated by a rectangular Gaussian spectrum centered at zero frequency and obeying the conditions listed in Section 5.1. The modulated signal can be expressed as

$$v(t)_{\phi_M} = A_0 \cos [\omega_c t + \phi + \phi(t)] \quad (8-1)$$

where $\phi(t)$ is the instantaneous phase-angle shift created by the modulating voltage. For phase modulation, PM, the information, $x(t)$, is directly proportional to $\phi(t)$, i.e.,

$$\phi(t) = k_1 x(t)$$

For frequency modulation, FM, the information is proportional to the instantaneous frequency. The only essential difference between the spectrum of the PM and the FM wave is a factor of ω^2 . The PM spectrum resulting from a modulating spectrum of $S_v(\omega)$ is identical to the spectrum resulting from FM with a modulating spectrum of $\omega^2 S_v(\omega)$ ¹⁴.

8.1 AUTOCORRELATION FUNCTION OF ϕ_M WAVE

The autocorrelation function of $v(t)_{\phi_M}$ is given by

$$R_v(\tau)_{\phi_M} = E \{ v(t)_{\phi_M} v^*(t + \tau)_{\phi_M} \}$$

From Eq. (2-30), this is seen to be

$$R_v(\tau)_{\phi_M} = \frac{A_0^2}{2} \operatorname{Re} E \left\{ e^{j[\phi(t) - \phi(t+\tau)]} \right\} \cos \omega_c \tau \quad (8-2)$$

The expectation value in Eq. (8-2) is

$$E \left\{ e^{j[\phi_1 - \phi_2]} \right\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\phi_1 - j\phi_2} p(\phi_1, \phi_2) d\phi_1 d\phi_2 \quad (8-3)$$

¹⁴Stewart, (REF. 13, p. 1540).

where

$$\phi_1 = \phi(t) \quad \text{and} \quad \phi_2 = \phi(t + \tau)$$

Comparing this expectation value with the joint characteristic function expression, Eq. (2-14), it is seen to be the particular characteristic function with $\xi_1 = 1$ and $\xi_2 = -1$, i.e.,

$$M_{xy}(j, -j) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{[jx - jy]} p(x, y) dx dy \quad (8-4)$$

Therefore, Eq. (8-2) can be written as

$$R_v(\tau)_{\phi M} = \frac{A_0^2}{2} \operatorname{Re} [M_{\phi_1 \phi_2}(j, -j)] \cos \omega_c \tau$$

Since $\phi(t)$ is described by a Gaussian random process with zero mean and variance σ_ϕ^2 , its joint characteristic function is, from Eq. (5-15),

$$M_{\phi_1 \phi_2}(j, -j) = e^{-\frac{1}{2} [\sigma_\phi^2 + \sigma_\phi^2 - 2R_\phi(\tau)]} = e^{-\sigma_\phi^2 + R_\phi(\tau)} \quad (8-6)$$

Therefore

$$R_v(\tau)_{\phi M} = \frac{A_0^2}{2} e^{-\sigma_\phi^2} e^{R_\phi(\tau)} \cos \omega_c \tau \quad (8-7)$$

8.2 POWER SPECTRUM OF ϕM WAVE FOR $\sigma_\phi^2 \ll 1$

The determination of the angle-modulated carrier spectrum from, Eq. (8-7) requires the Fourier transform of $e^{R_\phi(\tau)}$, which, in general, is quite difficult to obtain. For $\sigma_\phi^2 \ll 1$, however, it is well known that the spectrum of the angle modulated wave is similar to that of an amplitude modulated wave with the same modulating voltage¹⁵.

¹⁵ Abramson, (REF. 1, p. 411), Middleton, (REF. 6, p. 617) or Schwartz, (REF. 11, p. 118).

This can be shown easily by expanding the exponentials of Eq. (8-7),

$$R_v(\tau)_{\phi M} = \frac{A_0^2}{2} \left[1 - \sigma_\phi^2 + \frac{\sigma_\phi^4}{2!} - \dots \right] \left[1 + R_\phi(\tau) + \frac{R_\phi^2(\tau)}{2!} + \dots \right] \cos \omega_c \tau$$

For $\sigma_\phi^2 \ll 1$,

$$R_v(\tau)_{\phi M} \approx \frac{A_0^2}{2} [1 + R_\phi(\tau)] \cos \omega_c \tau \quad (8-8)$$

Therefore

$$S_v(f)_{\phi M} = \frac{A_0^2}{4} \delta(f \pm f_c) + \frac{A_0^2}{4} S_\phi(f) * \delta(f \pm f_c)$$

or

$$S_v(f)_{\phi M} = \frac{A_0^2}{2} \delta(f \pm f_c) + \frac{A_0^2}{4} S_\phi(f \pm f_c) \quad (8-9)$$

Comparing this with Eq. (7-5) gives the expected result that the spectrum of a narrowband ϕM carrier is similar to the spectrum of an AM carrier with the same modulating voltage.

8.3 CONVERTER OUTPUT SPECTRUM FOR ϕM INPUT

As done previously for the AM case in Section 7.2, the results of Section 6.3 can be utilized to obtain the output autocorrelation function and power spectral density of the converter.

The joint moments of $v(t)_{\phi M}$ are given by the values of Table 6-2, with the following substitutions,

$$\sigma_N^2 = \frac{1}{2} A_0^2 a_0 b \quad (8-10)$$

$$R_N(\tau) = \frac{1}{2} A_0^2 a_0 b \frac{\sin 2\pi \frac{b}{2} \tau}{2\pi \frac{b}{2} \tau} \cos \omega_c \tau \quad (8-11)$$

Utilizing the above substitutions, the converter output autocorrelation function for a narrowband ϕM carrier input is given by Eq. (6-11). The complete output spectrum for the converter is given by Eq. (6-13), with $S_N(f)$ replaced by the modulating spectrum of the ϕM wave, $S_\phi(f)$.

9. CONCLUSIONS

The output power spectral density of a diode frequency converter subjected to mixed statistical inputs has been derived by an application of the Wiener-Khinchine Theorem to the autocorrelation function of the diode output current. The series representation of the diode current is given by Eq. (3-17) and the complete output autocorrelation function is given by Eq. (4-5). The latter equation is useful for any input signal for which the statistical moments are known.

The output autocorrelation function for a signal plus additive noise input, again expressed in terms of the signal moments, is given by Eq. (5-21). The output spectrum for a cosine input is given by Eq. (6-10) and is displayed in Figure 6-2 for the 1N53C diode. The output spectrum for a cosine plus narrow-band noise input is given by Eq. (6-13), with the output signal-to-noise ratio given by Eq. (6-16). Sections 7 and 8 indicate the resulting spectra for an amplitude and an angle modulated signal plus noise input, respectively.

All of the autocorrelation and power spectra functions developed in this paper are expressed in series form and in terms of the diode conductance constants. This was done to permit a useful analysis of the contributions of each term and to indicate the significance of the diode constants to the resulting power spectra. From a display of the spectral contributions of the autocorrelation function, such as Figures (6-1) and (6-2), the terms producing undesired output noise and the diode terms affecting these terms are evident.

From these results, the optimum diode characteristic for efficient frequency conversion can be generated and the filtering and power requirements of the converter determined.

APPENDIX I

SPREADING RESISTANCE EFFECT ON CONDUCTANCE CONSTANTS

The current through the variable resistance diode with a spreading resistance of r_s was shown in Section 3.0 to be,

$$i(t) = I_0 \left[e^{a(v - iR_s)} - 1 \right] \quad (I-1)$$

where v and R_s are as shown in Figure 3-1.

Re-arranging terms,

$$\left[\frac{i}{I_0} + 1 \right] = e^{a(v - iR_s)} \quad (I-2)$$

Since,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad |x| < 1$$

therefore Eq. (I-2) can be expressed as,

$$a(v - iR_s) = \frac{i}{I_0} - \frac{1}{2} \frac{i^2}{I_0^2} + \frac{1}{3} \frac{i^3}{I_0^3} - \dots$$

Dividing by aR_s gives,

$$\left(\frac{v}{R_s} - i \right) = \frac{i}{aR_s I_0} - \frac{1}{2aR_s I_0^2} i^2 + \frac{1}{3aR_s I_0^3} i^3 - \dots$$

Re-arranging terms and letting $\frac{1}{aI_0} = r_d$,

$$\left(\frac{v}{R_s} \right) = \left(\frac{r_d}{R_s} + 1 \right) i - \left(\frac{r_d}{2I_0 R_s} \right) i^2 + \left(\frac{r_d}{3I_0^2 R_s} \right) i^3 - \dots \quad (I-3)$$

From Eq. (3-4),

$$i(t) = \sum_{k=1}^{\infty} g_k v^k$$

Expanding,

$$i = g_1 v + g_2 v^2 + g_3 v^3 + \dots$$

$$i^2 = g_1^2 v^2 + 2 g_1 g_2 v^3 + (2 g_1 g_3 + g_2^2) v^4 + \dots$$

$$i^3 = g_1^3 v^3 + 3 g_1 g_2 v^4 + 3 (g_1^2 g_3 + g_1 g_2^2) v^5 + \dots$$

Substituting the above series for the current into Eq. (1-3) and equating like powers of v gives,

$$\frac{1}{R_s} = g_1 \left(\frac{r_d}{R_s} + 1 \right) \quad (1-4)$$

$$0 = g_2 \left(\frac{r_d}{R_s} + 1 \right) - g_1^2 \left(2 I_0 \frac{r_d}{R_s} \right) \quad (1-5)$$

$$0 = g_3 \left(\frac{r_d}{R_s} + 1 \right) - 2 g_1 g_2 \left(\frac{r_d}{2 I_0 R_s} \right) + g_1^3 \left(\frac{r_d}{3 I_0^2 R_s} \right) \quad (1-6)$$

From Eq. (1-4),

$$g_1 = \frac{1}{R_s + r_d} \quad (1-7)$$

From Eq.'s (1-5) and (1-7),

$$g_2 = \frac{r_d}{2 I_0 (R_s + r_d)^3} \quad (1-8)$$

From Eq.'s (I-6), (I-7), and (I-8),

$$g_3 = \frac{r_d^2 - 2 r_d R_s}{6 I_0^2 (r_d + R_s)^5} \quad (I-9)$$

Using the same method of equating like powers in Eq. (I-3), the remaining conductance constants can be determined as required.

APPENDIX II

1N53 DIODE CHARACTERISTICS

In this appendix the diode parameters necessary for the frequency converter analysis will be determined from measurements on a representative microwave diode. The diode chosen for study was the 1N53C K_a-band microwave mixer, produced by Sylvania Electric Products, Inc. It is a point contact silicon diode in a miniature coaxial type package, and is designed for use as a first detector in K_a-band microwave superheterodyne receivers. The electrical characteristics, taken from the manufacturers engineering data sheet, are listed in Figure II-1.

The forward and reverse characteristics of four sample diodes were obtained on a Tektronix Type 575 Transistor-Curve Tracer, and from the four resulting curves a representative forward characteristic was plotted. This curve is shown as the solid line of Figure II-2.

Various values of α and R_s were assumed to obtain a 'best fit' curve for Eq. (I-1). I_0 was determined by an exact fit of the curve at the operating point chosen at

$$V_a = 0.25 \text{ v}, \quad I_a = 0.48 \text{ ma}$$

and by utilizing the relation,

$$I_0 = \frac{I_a}{e^{\alpha(V_a - I_a R_s)} - 1}$$

Several of the calculated curves are shown in Figure II-2. The curve that resulted in the best fit around the operating point, to three significant figures, had the values,

$$\alpha = 16 \text{ v}^{-1}$$

$$R_s = 25 \text{ ohms}$$

$$I_0 = 0.932 \times 10^{-5} \text{ amperes}$$

$$r_d = \frac{1}{\alpha I_0} = 6720 \text{ ohms}$$

DYNAMIC CHARACTERISTICS AT 25 C

	<u>Min.</u>	<u>Max.</u>
Conversion Loss, L_c , in db	—	6.5
Noise Ration, N_r	—	2.0
IF Impedance, Z_{IF} , in ohms	400	800
RF Impedance, Z_{rf} , as VSWR		
@ 34,860 Mc	—	1.6
@ 32,770 Mc	—	2.5
@ 36,950 Mc	—	2.5
Overall Noise Figure, NF, in db	—	9.0

TEST CONDITIONS:

For L_c : $f = 34,860$ Mc
 $P = 1.0$ mw
 $R_L = 100$ ohms
 $Z_m = 500$ ohms

For Z_{IF} : $f = 34,860$ Mc
 $P/I = 0.5$ ma dc

 For Z_{rf} : $P/I = 0.5$ ma dc
 $R_1 = 100$ ohms

For N_r : $f = 9,375$ Mc
 $P/I = 0.5$ ma dc (min)
 $R_L = 100$ ohms
 $R_c = 300$ ohms

For NF : $NF = L_c (N_{IF} + N_r - 1)$
 $N_{IF} = 1.5$ db

Figure II-1. Electrical Characteristics of IN53C Diode

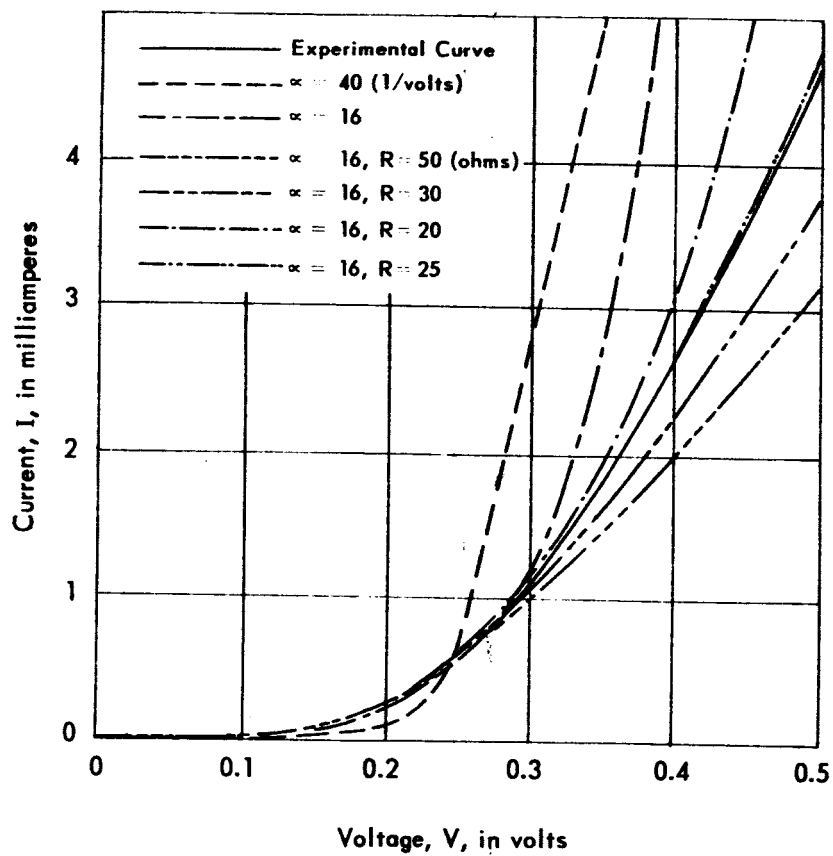


Figure II-2. 1N53C Current-voltage Characteristics

Since $r_d \gg R_s$, negligible error will be introduced by neglecting R_s and utilizing Eq. (3-5) to determine g_1, g_2 , etc., instead of Eq.'s (I-7), (I-8), and (I-9). The first six conductance constants for the 1N53C diode, along with the other important constants for the diode, are summarized in Figure II-3.

The 'A' coefficients for the diode, and their squares, are shown in Figure II-4. They were determined from Eq. (3-16), with the local oscillator voltage amplitude, E_0 , set equal to 0.25 volts.

EXPONENTIAL REPRESENTATION

$$i = I_0 \left[e^{\alpha(v - iR_s)} \right]$$

$$I_0 = 0.932 \times 10^{-5} \text{ a}$$

$$\alpha = 16 \text{ v}^{-1}$$

$$R_s = 25 \text{ } \Omega$$

POWER SERIES REPRESENTATION

$$i = \sum_{k=0}^{\infty} g_k v^k \quad g_k \text{ in } \Omega^{-k}$$

$$g_0 = 0$$

$$g_1 = 1.48 \times 10^{-4}$$

$$g_2 = 1.18 \times 10^{-3}$$

$$g_3 = 6.16 \times 10^{-3}$$

$$g_4 = 2.46 \times 10^{-2}$$

$$g_5 = 8.60 \times 10^{-2}$$

$$g_6 = 2.30 \times 10^{-1}$$

Figure II-3. Experimentally Determined Constants for the 1N53C Diode

$A_{00} = 0.0369 \text{ ma}$	$A_{00}^2 = 0.00136 \text{ ma}^2$
$A_{01} = 0.109 \text{ ma}$	$A_{01}^2 = 0.0119 \text{ ma}^2$
$A_{02} = 0.0369 \text{ ma}$	$A_{02}^2 = 0.00136 \text{ ma}^2$
$A_{03} = 0.012 \text{ ma}$	$A_{03}^2 = 0.000144 \text{ ma}^2$
$A_{10} = 0.436 \text{ ma/v}$	$A_{10}^2 = 0.19 \text{ ma}^2/\text{v}^2$
$A_{20} = 5.79 \text{ ma/v}^2$	$A_{20}^2 = 33.5 \text{ ma}^2/\text{v}^4$
$A_{30} = 19.6 \text{ ma/v}^3$	$A_{30}^2 = 384.0 \text{ ma}^2/\text{v}^6$
$A_{11} = 1.74 \text{ ma/v}$	$A_{11}^2 = 3.03 \text{ ma}^2/\text{v}^2$
$A_{12} = 0.578 \text{ ma/v}$	$A_{12}^2 = 0.334 \text{ ma}^2/\text{v}^2$
$A_{13} = 0.192 \text{ ma/v}$	$A_{13}^2 = 0.0369 \text{ ma}^2/\text{v}^2$
$A_{21} = 11.6 \text{ ma/v}^2$	$A_{21}^2 = 135.0 \text{ ma}^2/\text{v}^4$
$A_{22} = 4.62 \text{ ma/v}^2$	$A_{22}^2 = 21.3 \text{ ma}^2/\text{v}^4$
$A_{23} = 1.68 \text{ ma/v}^2$	$A_{23}^2 = 2.82 \text{ ma}^2/\text{v}^4$
$A_{31} = 78.6 \text{ ma/v}^3$	$A_{31}^2 = 6180 \text{ ma}^2/\text{v}^6$
$A_{32} = 26.9 \text{ ma/v}^3$	$A_{32}^2 = 723 \text{ ma}^2/\text{v}^6$
$A_{33} = 8.9 \text{ ma/v}^3$	$A_{33}^2 = 79.2 \text{ ma}^2/\text{v}^6$

Figure II-4. 'A' Coefficients for the IN53C Diode, with $E_0 = 0.25$ Volts

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